Bachelor Internship

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Distance Between Modal Interpretations for Belief Revision

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Fiche synthèse :
Distance entre interpretations modales pour la révision des croyances

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Contexte général

Un système bénéficiant de connaissances peut-être amené à apprendre des informations contradictoires. La révision des croyances vise à rétablir la cohérence dans les connaissances d'un système, en préservant un maximum d'informations. La révision des croyances en logique classique est un domaine établie. Elle fut introduite par le travail de Carlos E Alchourrón, Peter Gärdenfors et David Makinson en 1985. Les opérateurs introduits dans leur travail présentent un défaut de cohérence lors de révisions répétitives. En 2001, Daniel Lehmann, Menachem Magidor et Karl Schlechta proposent de baser les révisions des croyances sur des distances entre valuations, ce qui rétabli une cohérence entre révisions successives.

D'autre part, la logique modale est une logique plus riche que la logique classique, qui introduit les notions de possibilité, impossibilité, nécéssité et contingence. Cette nouvelle logique s'interprète non plus avec des valuations mais avec des modèles de Kripke. Étendre la révision des croyances à la logique modale permettrait à des systèmes plus complexes et aux connaissances plus nuancées de mettre à jour leur connaissances automatiquement en préservant leur cohérence.

Le problème étudié

Pour parvenir à cet objectif, le principal obstacle est de définir une distance entre modèles de Kripke, qui tiendrait le même rôle que la distance entre valuations dans le processus classique. Un tel travail a été réalisé par Thomas Caridroit en 2016, mais dans le cas très spécifique des modèles de Kripke Kd45n. Par ailleurs, d'autres travaux s'intéressent à la révision des croyances en logique modale sans passer par des distances.

La contribution proposée

Mon travail durant ce stage a été de déterminer plusieurs distances entre modèles de Kripke et d'étudier leurs propriétés. Pour y parvenir, j'ai décomposé le problème en plusieurs étapes. J'ai donc défini des méthodes pour construire des distances entre ensembles, entre n-uplets, entre relations et entre fonctions. En combinant toutes ces méthodes, j'ai défini une grande quantité de distances entre modèles de Kripke. J'ai ensuite étudié leurs propriétés, ainsi que celles de distances préexistantes.

L'étude des propriété des distances reprend la forme d'étude axiomatique introduite par Carlos E Alchourrón, Peter Gärdenfors et David Makinson. Comme eux, je suis parvenue à caractériser deux de mes distances entre ensembles par les axiomes qu'elles satisfont.

J'ai aussi prouvé qu'une propriété fort désirable pour une distance entre ensembles était impossible à satisfaire (l'axiome CR4).

Les arguments en faveur de sa validité

La pertinence des distances proposée est encore à déterminer. Une doctorant poursuivra ce travail à partir de septembre 2022. Cependant, les méthodes que j'ai définies sont très générales et proposent une grande variété de choix pour construire des distances entre modèles de Kripke. Le travail restant est donc de déterminer quelles propriétés sont désirables pour une distance à utiliser en révision des croyances basées sur une distance. Ensuite, mon étude axiomatique permettra de choisir la meilleure construction. Par ailleurs, ces méthodes permettent de construire des distances entre de nombreux objets mathématiques, ce qui peut servir pour des domaines très différents. Ils peuvent notamment s'appliquer pour la révision des croyances basées sur une distance pour d'autres logiques.

Le bilan et les perspectives

La prochaine étape est de déterminer quelles sont les propriétés souhaitables pour une distance en révision des croyances. Ensuite, mon étude permettra de choisir parmi les constructions que je propose. Un autre point reste à résoudre : la traduction entre un ensemble de formules et l'ensembles des modèles de Kripke qui satisfont ces formules. Cette traduction semble possible mais d'une très grande complexité calculatoire. Il faut aussi trouver un moyen de rendre fini l'ensemble des formules correspondant à un ensemble de modèles de Kripke. Par exemple, si la formule φ est vraie, alors $\varphi \wedge \varphi$ l'est aussi, ainsi que $\varphi \wedge \varphi \wedge \varphi$, etc. Ce point est donc encore à travailler, en trouvant une bonne façon de quotienter ces ensembles.

Finalement, ma contribution a été de proposer tout un ensemble de méthodes pour construire et étudier des distances entre toutes sortes d'objets mathématiques.

J'ai aussi défini ma propre distance entre ensembles, caractérisé une distance et une fonction, et réalisé une étude complète des propriété de plusieurs fonctions qui semblent pertinentes pour la révision des croyances basées sur une distance.

Le résultat d'impossibilité que j'ai prouvé annonce des fortes et inattendues limites sur la précision des distances que l'on pourra utiliser.

Bachelor internship report

Distance between modal interpretations for belief revision

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Contents

1 Introduction

Distance-based belief revision, introduced by [LMS01], is an established field in classical logic. The purpose of my internship is to extend this method to modal logic. The global process is mostly the same, but we now need a distance between modal interpretations to replace the distance between valuations used in classical distance-based belief revision. During my internship, I introduced useful methods to construct distances between composed objects, such as modal interpretations and Kripke models. I also studied axiomatically these constructed distances and I characterized some of them.

This section introduces modal logic and classical belief revision and defines the goal of the internship. Section 2 introduces axiomatic reasonings and the axioms that I studied during my internship. Section 3 establishes distances and pre-distances and their axiomatic study. Section 4 applies my work to modal belief revision.

In the present document, $\star$ indicates propositions that I proved and definitions that I introduced during my internship and $\diamond$ indicates propositions that I proved but that were already proved in the literature. Only my most important results are in this report, the others are to be found in the appendix, as well as all the proofs.

1.1 Modal logic

In classical logic, a proposition can be either true or false. Its truth value can possibly be undefined, but as soon as a proposition has a truth value, it is limited to true or false. Modal logic is a logic in which truth values belong to a wider set of values, for example: possible, impossible, necessary and contingent. Formally, propositions are still true and false in modal logic, but the language contains new operators corresponding to modalities. These modalities depend on the modal logic which is considered. In this report, we consider the modal logic with the operator $\Box$ called "box" and expressing necessity.

We consider the language $\mathtt{L}$ generated by $\langle\ \mathtt{A}, \wedge, \neg, \Box, \top, \bot\ \rangle$, where $\mathtt{A}$ is a finite set of atomic propositions and where $\top$ stands for "true" and $\bot$ stands for "false". The same way as we can define $\vee$ using $\neg$ and $\wedge$, we can define the possibility operator $\Diamond$ using $\neg$ and $\Box$: let φ be a formula of $\mathtt{L}$, then $\Diamond\varphi$ is equivalent to $\neg\Box\neg\varphi$. (φ is possible if and only if it is not necessary that φ is false).

A possible way to evaluate propositions of $\mathtt{L}$ is by using Kripke models. First, we define classical valuations:

Definition 1. A *valuation* on $\mathtt{A}$ is a function from $\mathtt{A}$ to $\{0,1\}$. The set of all valuations on $\mathtt{A}$ is noted by $\mathtt{Val}(\mathtt{A}) \triangleq \{0,1\}^{\mathtt{A}}$.

Definition 2. A *Kripke model* is a triple $\langle W, R, f\rangle$ where:

- W is a non empty set. Its elements are called "worlds".
- R is a relation on W called *accessibility relation.*
- $f : W \to \mathtt{Val}(\mathtt{A})$ is called *interpretation function* or *labelling function*

Then we define how to evaluate the truth value of a modal proposition with a Kripke model:

Definition 3. We define recursively the truth of a formula of $\mathtt{L}$ in a world w of a model $\langle W, R, f\rangle$ by:

$$\begin{array}{lll} w \models \top,\ w \not\models \bot & & \\ w \models a & \text{if and only if} & f(w)(a) = 1 \text{ when } a \in \mathtt{A} \\ w \models \neg\varphi & \text{if and only if} & w \not\models \varphi \\ w \models \varphi \wedge \psi & \text{if and only if} & w \models \varphi \text{ and } w \models \psi \\ w \models \Box\varphi & \text{if and only if} & \forall w' \in W, wRw' \Rightarrow w' \models \varphi \end{array}$$

A formula φ of $\mathtt{L}$ is true in a model $\langle W, R, f\rangle$ if $\forall w \in W, w \models \varphi$.

1.2 Belief revision

Any intelligent system able to collect new information can have to face a situation where his knowledges are contradictory. The purpose of belief revision is to find an efficient algorithm allowing such systems to correct its knowledges, in order to keep them consistent. Several questions appear while trying to solve this problem: How to know which knowledge to keep ? How to keep as many knowledges as possible during revision ? How to ensure that consecutive revisions will stay relevant ? [AGM85] introduced abstract revision operators. [LMS01] then introduced the idea that revision could be done using a distance between valuations, this is the distance-based belief revision. [DP97] exhibited weaknesses of the revision operators defined by [AGM85] in the case of iterated belief revision. In the distance-based belief revisions, the distance gives a coherence to the successive revisions which could be the solution for successful iterated belief revisions. [Hub13] summerises the AGM theory and several works about iterated belief revision.

1.3 Distance-based belief revision

A valuation can be understood as a description of the state of the world, through the truth value of the considered variables. Then, assuming that a set of formulas is true, we know that there are some worlds in which we can be, and others in which we cannot be. This set of possible worlds, in fact a set of valuations, is a mathematical object that we can study with distances. This set furthermore countains exactly the same abstract information as the original set of formulas. The idea of distance-based belief revision is to translate the set of formulas representing the beliefs of the system, into the semantic world of valuations. Formally, if Γ is the set of beliefs at the begining and α is the new information, we translate them in sets of valuations with the following operators. In our case, we assume that α is more reliable than Γ. Then we consider the set of valuations compatible with α which are closest to the valuations compatible with Γ, and finally, we translate it back to a set of formulas with $\mathtt{Th}$:

Definition 4. We assume a language $\mathtt{L}'$ of classical logic, and X be a set of valuations on $\mathtt{L}'$, we define:
$\mathtt{Th}_{\mathtt{L}'}(X) \triangleq \{\alpha \in \mathtt{L}' \mid X \models \alpha\}$ (The set of all the formulas of $\mathtt{L}'$ satisfied by all valuations of X)
Let $\alpha \in \mathtt{L}'$, $\mathtt{Mod}(\alpha) \triangleq \{v \in \mathtt{Val}(\mathtt{A}) \mid v \models \alpha\}$. (The set of all valuations satisfying (modeling) α)
Let $\Gamma \in \mathtt{Pow}(\mathtt{L}'), \mathtt{SetMod}(\Gamma) \triangleq \{v \in \mathtt{Val}(\mathtt{A}) \mid v \models \Gamma\}$ (where $v \models \Gamma$ means $\forall \varphi \in \Gamma, v \models \varphi$).

Definition 5. A *revision operator* is a function $* : \mathtt{Pow}(\mathtt{L}') \times \mathtt{L}' \to \mathtt{Pow}(\mathtt{L}')$. A *distance-based revision* d is a revision operator such that, for all $\langle \Gamma, \alpha \rangle \in \mathtt{Pow}(\mathtt{L}') \times \mathtt{L}'$: $\Gamma * \alpha \triangleq \mathtt{Th}_{\mathtt{L}'}(\ \mathtt{Closest}(\mathtt{Mod}(\alpha), \mathtt{SetMod}(\Gamma))\ )$ where $\mathtt{Closest}(\mathtt{Mod}(\alpha), \mathtt{SetMod}(\Gamma))$ is the set of the elements of $\mathtt{Mod}(\alpha)$ the closest to $\mathtt{SetMod}(\Gamma)$ for a distance d between valuations and set of valuations.

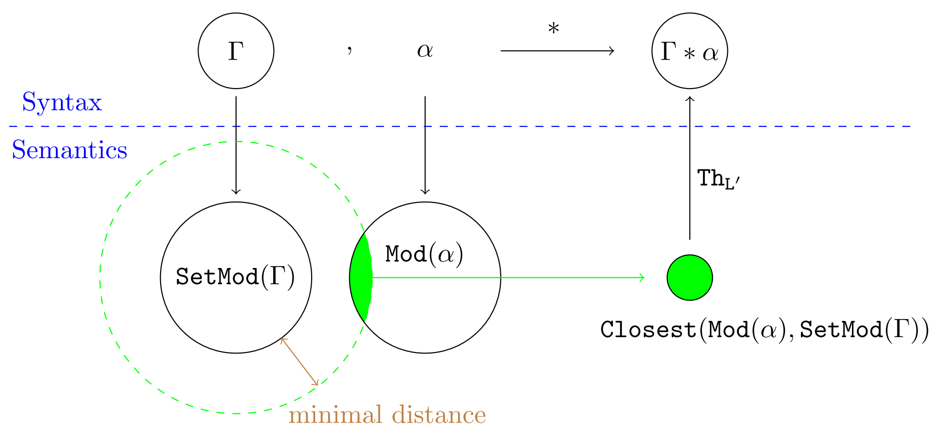


Figure 1: Process of the distance-based revision

1.4 Goal of the internship

The goal of my internship was to generalise distance-based belief revision to modal logic. More precisely, I had to find a distance between Kripke models, and if possible to find the interresting properties it satisfies. Once such a distance is defined, the same process as in classical distance-based belief revision can be executed, replacing the distance between valuations by a distance between Kripke models.

Because a Kripke model is a triple of a set, a relation and a function, and because a distance between triples can be constructed by adding the distances between each element, I decided to divide my work in two parts: first I defined distances between tuples, between sets, between relations and between functions, and secondly I put all of these distances together to construct distances between Kripke models. I also studied the distances axiomatically, as explained in Section 2.

A similar work was realised in [CKdLM16] about the specific case of KD45n Kripke models. In [GW19], two characterizations for revision operators were exhibited. These revisions, which also use Kripke models, are not distance-based, but the second characterization uses a similar intuition by counting the necessary modifications to get from one model to another.

2 Requirements on distances

The usual properties of a distance might not be all relevant, so we define the notion of pre-distance, which is more general:

Definition 6. Let X be a set. A *pre-distance* on X is a function from $X \times X$ to $[0, +\infty)$.

For example, in the mountains, it can be more difficult to go from a point A to a point B than the other way round if A is higher than B. Thus symmetry is not always a relevant property. It can also cost energy to stay in a given state, in which case respect of identity is not relevant.

2.1 Axiomatic reasonings

Axiomatic reasonings were first introduced by [AGM85] and widely reused afterwards. The idea is to characterize a function with "axioms", which are in fact properties that the function satisfies. The axiomatic study allows to better understand a function and to know whether this function is relevant for some purpose or not. [AGM85] used this method to characterize his definition of revision.
The book [Gär88] details the axiomatic characterization of the expansion, the contraction and the revision operators.

2.2 Axioms on pre-distances

We then express the properties of a pre-distance through axioms. Each axiom expresses whether a particular information about the similarities and differences between the two elements in parameter is conveyed or not by the pre-distance F.

2.2.1 Axioms on pre-distances

Definition 7 ($\star$)**.** Let X be a set and F be a pre-distance on X. We define the following axioms:

(CR1) : (*Identity respecting*) $\forall A, B \in X,\ F(A,B) = 0 \iff A = B$

(CR2) : (*Symmetric*) $\forall A, B \in X,\ F(A,B) = F(B,A)$

(CR3) : (*Triangle inequality*) $\forall A, B, C \in X,\ F(A,B) \leqslant F(A,C) + F(C,B)$

(CR1$_\mathtt{W}$) : (*Weak respect of identity*) $\forall A \in X,\ F(A,A) = 0$

The usual definition of a distance now appears as a specific pre-distance:

Definition 8. A *distance* is a pre-distance that satisfies (CR1), (CR2) and (CR3).

2.2.2 Axioms on pre-distances between sets

We introduce the following notation:

Notation :

- We note by $\#X$ the cardinal of a set X.
- Let X be a set of sets, we note by $\bigcup X$ the union of all elements of X, ie. $\bigcup X \triangleq \bigcup\limits_{A \in X} A$.

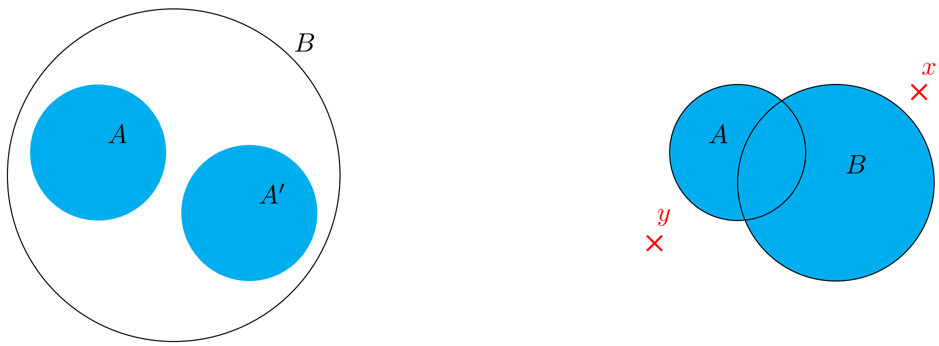
  We use as convention: $\bigcup \emptyset = \emptyset$.

Definition 9 ($\star$)**.** Let X be a set of sets and F a pre-distance on X. We define the following axiom:

(CR5) : (*Subset's equivalence*)
$\forall A, A', B \in X$, $(A \subseteq B \wedge A' \subseteq B \wedge \#A = \#A') \Rightarrow F(A, B) = F(A', B)$

(CR6) : (*Stranger's equivalence*)
$\forall A, B \in X$, $\forall x, y \in \bigcup X$, $\{x, y\} \cap (A \cup B) = \emptyset \Rightarrow F(A \cup \{x\}, B) = F(A \cup \{y\}, B)$

(CR5) means that two subsets of a same set, with the same cardinal, are treated the same way by F. That is to say that the identity of the elements of a subset does not matter. Only the quantity of elements in the subset can affect F.
(CR6) axiom expresses the anonymity of extern elements. If an element is new to A and B, then adding it to A or adding another new element to A has the same effect on the difference between this union and B, defined with F.



$F(A, B) = F(A', B)$ $\qquad$ $F(A \cup \{x\}, B) = F(A \cup \{y\}, B)$

(a) (CR5): (A and A' are equivalent for F when compared to B)

(b) (CR6): (Any extern element to A and B is equivalent for F)

Figure 2: Illustration of (CR5) and (CR6)

Definition 10 ($\star$)**.** Let X be a set of sets and F a pre-distance on X. We define the following axioms:

(CR7$_{\texttt{Str}}$) : (*Stranger decomposability*)
$\forall A, B \in X$, $\forall x \in \bigcup X$, $\{x\} \cap (A \cup B) = \emptyset \Rightarrow F(A \cup \{x\}, B) = F(A, B) + F(\{x\}, B)$

Let K be a real number,
(CR8$_K$) : (*K-countability*)
$\forall A, B \in X$, $\forall x \in \bigcup X$, $\{x\} \cap (A \cup B) = \emptyset \Rightarrow F(A \cup \{x\}, B) = F(A, B) + K$

(CR16) : (*Inserting growth*)
$\forall A, B \in X$, $\forall x \in \bigcup X$, $\{x\} \cap (A \cup B) = \emptyset \Rightarrow F(A \cup \{x\}, B) > F(A, B)$

(CR7) expresses the fact that the pre-distance F can be decomposed in the distances between each element of A stranger to B and B. (CR8$_K$) expresses the fact that each element is counted with the same value: 1. (CR16) ensures that adding a new element to a set A will increase the difference between A and B expressed by $F(A, B)$, as soon as the new element is not in B.

2.2.3 Axioms on pre-distances between subsets of a finite metric space

We need a number majoring all the possible distances in a finite metric space:

Definition 11 ($\star$)**.** Let $\langle X, d\rangle$ be a finite metric space, we define $\texttt{Dmax}(X, d) \triangleq \max_{(x,y)\in X^2} d(x, y)$.

Definition 12 ($\star$)**.** Let $\langle X, d\rangle$ be a finite metric space. We define the following axioms:

(d-CR4) : (*Elementary d-monotony*)

$$\forall (x,x',y,y') \in X^4,\ \forall A \subseteq X, \forall B \subseteq X,$$
$$(d(x,x') < d(y,y') \wedge \{x,y\} \cap A = \emptyset \wedge \{x',y'\} \cap B = \emptyset) \Rightarrow F(A \cup \{x\}, B \cup \{x'\}) < F(A \cup \{y\}, B \cup \{y'\})$$

(d-CR11) : (*d-represented*)
$\forall A \subseteq X, \forall B \subseteq X,\ (A \neq \emptyset \wedge B \neq \emptyset) \Rightarrow \exists (a,b) \in A \times B \mid F(A,B) = d(a,b)$

(d-CR17a) : (*d-singleton fidelity*)
$\forall a, b \in X,\ F(\{a\},\{b\}) = d(a,b)$

(d-CR4) expresses the capacity of a pre-distance F to express the distances between each element of the sets A and B. If the distance between two elemens of A and B increses, (d-CR4) ensures that $F(A,B)$ also increases. We see in subsection 2.4 that this axiom is problematic.

2.2.4 Axioms on pre-distances on cartesian products

Definition 13 ($\star$)**.** Let $\langle X, d\rangle$ be a finite metric space and Y be a set. We define:

(d-CR12$_\mathtt{L}$) / (d-CR12$_\mathtt{R}$) / (d-CR12$_n$): (*d-left-monotony, d-right-monotony, d-n-monotony* (for tuples)).
(*Left-monotony*): $\forall (x,x',y,y') \in X^4,\ \forall (a,b) \in Y^2,\ d(x,x') < d(y,y') \Rightarrow F(\langle x,a\rangle, \langle x',b\rangle) < F(\langle y,a\rangle, \langle y',b\rangle)$
(d-right-monotony is the same rights, and n-d-monotony is the same at rank n.)

This axiom allows us to know the properties of d are preserved through the combination of several pre-distances introduced in Substections 3.1 and 3.4 and in Section 4.

2.3 Relations beween axioms

It is worth noticing that one pre-distance cannot satisfy all the axioms. Therefore, I study in this subsection the relations of compatibility and incompatibility of the axioms. I also examine dependences and independences between axioms. This gives us an overview of which combinations of axioms is possible or impossible to reach with a given distance. For example, (CR4) is a very desirable axiom that cannot be reached in a large amount of finite metric spaces, as we will see.

Proposition 1 ($\star$)**.** *We have the following relations between axioms:*
Implications:

- (CR1) + (CR2) + (CR8) $\Rightarrow$ (CR3), (CR5), (CR6)
- (CR8) $\Rightarrow$ (CR6), (CR16)
- (d-CR11) $\Rightarrow$ (d-CR17a)
- (CR7$_\mathtt{Str}$) $\Rightarrow \forall x \in X,\ F(\emptyset, \{x\}) = 0$

Independences:

- (CR6) + (CR16) $\not\Rightarrow$ (CR8)
- (CR2) + (CR8) $\not\Rightarrow$ (CR1)
- (d-CR17a) $\not\Rightarrow$ (d-CR11)

Incompatibilities:

- (CR7$_\mathtt{Str}$) $\Rightarrow \neg$(CR1)
- (CR1) + (CR2) + (CR8) $\Rightarrow \neg$(CR7$_\mathtt{Str}$)

We now know which properties are desirable for a pre-distance to satisfy, and we also know that we will have to make a choice at some point, because it is impossible to convey all the information through a same pre-distance. Knowing this, we will construct concrete pre-distances and examine which axioms they satisfy.

2.4 Impossibility result

We first define valuations and a distance between valuations:

Definition 14. Let X be a set, we define the *Hamming distance* $\texttt{Ham}$ as the distance on $\texttt{Val}(X)$ such that $\forall v, w \in \texttt{Val}(X)$: $\texttt{Ham}(v, w) \triangleq \#\{x \in X \mid v(x) \neq w(x)\}$

Proposition 2 ($\star$)**.** ($\texttt{Eucl}$-CR4) *cannot be satisfied (if* $\texttt{Eucl}$ *is the euclidian distance in a geometric space).* ($\texttt{Ham}$-CR4) *cannot be satisfied.*

Proof. (($\texttt{Eucl}$-CR4) *cannot be satisfied*) We consider the set $X = \{a, b, c, d, e\}$ on Figure 3. Suppose there exists a function F on $X \times X$ respecting ($\texttt{Eucl}$-CR4), then:
$\texttt{Eucl}(a, d) > \texttt{Eucl}(a, e)$ implies $F(\{b, a\}, \{c, d\}) > F(\{b, a\}, \{c, e\})$ (red in the figure)
$\texttt{Eucl}(b, e) > \texttt{Eucl}(b, d)$ implies $F(\{a, b\}, \{c, e\}) > F(\{a, b\}, \{c, d\})$ (green in the figure)
We find a contradiction, so no such F exists. □

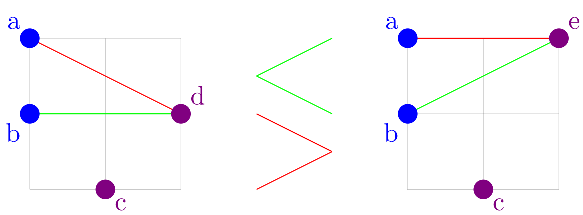


Figure 3: Counter-example for (CR4)

In fact the satisfiability of (d-CR4) strongly depends on X. A further study of the cases where (CR4) is impossible is to be found in the appendix.

3 Pre-distances and distances

3.1 Candidates

3.1.1 Most general distance

We first define the drastic distance, which can be used on any set.

Definition 15. Let X be a set. The *drastic distance* on X is the distance on X such that $\forall A, B \in X$: $\texttt{Drast}(A, B) \triangleq \begin{cases} 0 \text{ if } A = B \\ 1 \text{ otherwise} \end{cases}$

Thanks to $\texttt{Drast}$ we can make a metric space out of any set. $\texttt{Drast}$ is especially useful as a piece of construction when no distance on X is known.

3.1.2 Distances on cartesian products

Although being very simple, the following lemma proves to be central in the construction of distances of composed objects. Indeed, it allows us to decompose any complex problem into several elementary problems. It can be used to define pre-distances not only between cartesian products, but also between relations and even between functions, because these objects are pairs or are isomorphic to pairs.

Lemma 1 ($\star$)**.** *Let* $n \in \mathbb{N}$, $(\langle X_i, d_i \rangle)_{i \in [\![1,n]\!]}$ *be* n *metric spaces. We note* $\boldsymbol{d} \triangleq (d_i)_{i \in [\![1,n]\!]}$ *and* $\boldsymbol{X} \triangleq X_1 \times \ldots \times X_n$*, then the function* $\texttt{Sum}_{\boldsymbol{d}}$ *on* $\boldsymbol{X} \times \boldsymbol{X}$ *such that* $\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$: $\texttt{Sum}_{\boldsymbol{d}}(\boldsymbol{x}, \boldsymbol{y}) \triangleq \sum_{i=1}^{n} d_i(x_i, y_i)$ *is a distance on* $\boldsymbol{X}$.

3.1.3 Pre-distances between sets

Definition 16 ($\diamond$)**.** Let X be a set of finite sets. We define $\texttt{Delta}(A, B) \triangleq \#(A\Delta B)$ where $A\Delta B$ is the *symmetric difference* between A and B (ie. $A\Delta B \triangleq (A \setminus B) \cup (B \setminus A)$).

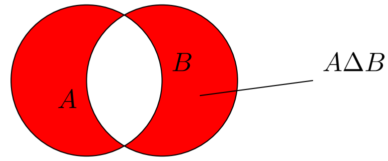


Figure 4: Symmetric difference

Proposition 3 ($\diamond$)**.** Delta *is a distance on any set of finite sets.*

Delta is exactly Ham applied to the indicator functions of the considered sets.

This result was already proven in [Bes73], which also defines other relevant distances between subsets of a set. The common drawback of all these distances is that they do not consider the distance between the elements of the subsets. We formalise this property in the axiomatic study of Delta.

Definition 17 ($\star$)**.** Let X be a finite set of finite sets. Then $\bigcup X$ is finite and we can label its elements: $\bigcup X = \{x_1, x_2, ...\}$ to create an order between them.
Let $A \in X$, $\mathbb{1}_A$ can be seen as $(\mathbb{1}_A(x_i))_{i\in[\![1,\#\bigcup X]\!]}$, which is a finite sequence of 0 and 1. This can be seen as a binary number.
We define $\texttt{Numb}(A)$ as the element of $\mathbb{N}$ that corresponds to the binary number representer by $(\mathbb{1}_A(x_i))_{i\in[\![1,\#\bigcup X]\!]}$.
Then we define: $\texttt{Bin}(A, B) \triangleq |\texttt{Numb}(A) - \texttt{Numb}(B)|$

Proposition 4 ($\star$)**.** Bin *is a distance on any finite set of finite sets.*

This distance seems to be very likely unfitted for distance-based belief revision, and we will formalise why in the axiomatic study.

The two previous distances have a strong drawback for belief revision: they do not consider the distance between the elements of the sets. For example, if our set of atomic propositions is $\{a, b, c\}$ and if we note the valuation v such that $v(a) = 0$, $v(b) = 1$, $v(c) = 1$ by $(0, 1, 1)$, and so on, then:

$$\texttt{Delta}(\{(0,0,0)\}, \{(0,0,1)\}) = 1 = \texttt{Delta}(\{(0,0,0)\}, \{(1,1,1)\})$$

while the two sets on the left seem closer to each other than the two on the right.

In the next subsection, we define distances and pre-distances which consider this matter. On the other hand, as one pre-distance cannot convey all the information, the pre-distances of the next section do not express the cardinality of the difference between two subsets.

3.1.4 Pre-distances between subsets of a metric space

The main difficulty to define a distance between Kripke semantics is to achieve to construct a distance between sets of elements, using a distance between the elements.

In the case of S5 Kripke semantics, the starting distance will be the Hamming distance. The pre-distances defined in this section will also be used to define pre-distances between relations, using as starting pre-distance a pre-distance between pairs, thanks to the observation that relations are sets of pairs.

In this subsection, I introduce several pre-distances and prove that only two of them are distances. One of them is the already known Hausdorff distance, noted by Haus.

We first use as convention $\min(\emptyset) = 0$; which is perfectly counter-intuitive but necessary here.

Definition 18. Let $\langle X, d\rangle$ be a metric space. We define the following functions on $\texttt{Pow}(X) \times \texttt{Pow}(X)$, such that $\forall A, B \in \texttt{Pow}(X)$: (with $\min(\emptyset) = 0$)

$\texttt{Haus}_d(A, B) \triangleq \max(\max_{b\in B}\min_{a\in A} d(a, b), \max_{a\in A}\min_{b\in B} d(a, b))$ (the *Hausdorff distance*)

Definition 19 ($\star$)**.** Let $\langle X, d\rangle$, let $A, B \in \mathtt{Pow}(X)$ and $f : A \to B$. We define:

$$\mathtt{Dist}(d, A, B, f) \triangleq \sum_{a\in A} d(a, f(a))$$

Definition 20 ($\star$)**.** Let $\langle X, d\rangle$ be a metric space. We define the following functions on $\mathtt{Pow}(X) \times \mathtt{Pow}(X)$, such that $\forall A, B \in \mathtt{Pow}(X)$: (with $\min(\emptyset) = 0$)

$$\mathtt{Inj}_d(A, B) \triangleq \min_{f:A\hookrightarrow B} \mathtt{Dist}(d, A, B, f) + \min_{f:B\hookrightarrow A} \mathtt{Dist}(d, B, A, f) + |\#A - \#B| \times \mathtt{Dmax}(X, d)$$

We then use as convention $\min(\emptyset) = \mathtt{Dmax}(X, d)$ for the next pre-distances.

Definition 21 ($\star$)**.** Let $\langle X, d\rangle$ be a metric space. We define the following functions on $\mathtt{Pow}(X) \times \mathtt{Pow}(X)$, such that $\forall A, B \in \mathtt{Pow}(X)$: (with $\min(\emptyset) = \mathtt{Dmax}(X, d)$)

- $\mathtt{PointSet}_d(A, B) \triangleq \sum_{x\in A} \mathtt{dist}(x, B, d) + \sum_{y\in B} \mathtt{dist}(y, A, d)$

  where $\mathtt{dist}(x, B, d) \triangleq \min\{d(x, y)|y \in A\}$
- $\mathtt{Pairs}_d(A, B) \triangleq \sum_{(x,y)\in A\times B} d(x, y) \;-\; \sum_{(x,y)\in(A\cap B)^2} d(x, y)$

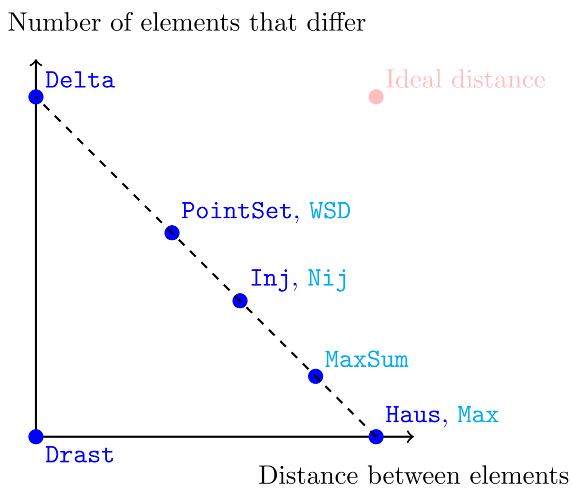


Figure 5: Expressivity of the pre-distances

Proposition 5 ($\star$)**.** *Let $\langle X, d\rangle$ be a metric space, $\mathtt{Inj}_d$ and $\mathtt{Haus}_d$ are distances between the subsets of X, and $\mathtt{PointSet}_d, \mathtt{Pairs}_d$ are not.*

Finally, Figure 5 summerizes which pre-distances express the cardinality of the difference between sets, and which ones consider the distances between the elements, and in which proportions. The pre-distances in cyan are defined and studied in the appendix.

3.1.5 Distances between relations

This subsection is a direct application of the previous results. Let X be a set. A relation on X is no more than a set of pairs of elements of X. Suppose we already know a distance d on X. Then by lemma 1, $\mathtt{Sum}_{(d,d)}$ is a distance on $X \times X$, ie., a distance between pairs of X. Then $\mathtt{Haus}_{\mathtt{Sum}(d,d)}$ is a distance on $\mathtt{Pow}(X \times X)$, ie., a distance between relations on X. Similarly we have:

Proposition 6 ($\star$)**.** *Let $\langle X, d\rangle$ be a metric space, $\mathtt{Haus}_{\mathtt{Sum}(d,d)}$ and $\mathtt{Inj}_{\mathtt{Sum}(d,d)}$ are distances between relations on X.*

The same can be done with other pre-distances, which could result in pre-distances that are not distances, but still are interresting.

3.1.6 Distances between functions

In a similar way, we apply the previous results. Let X, Y be two sets. A function $f : X \to Y$ can be seen as the subset of $X \times Y$, of all pairs $\langle x, f(x)\rangle$. More formally:

Definition 22. Let X, Y be two sets. Let $f : X \to Y$ be a function, we define the *graph* of f as

$$\texttt{Graph}(f) \triangleq \{\langle x, f(x)\rangle \mid x \in X\}.$$

A first option is to use a distance between sets, such that $\texttt{Delta}$, to compare the graphs of the functions. More formally:

Definition 23 ($\star$)**.** Let X be a finite set and Y be a set, we define the function on $Y^X \times Y^X$ such that $\forall f, g \in Y^X$: $\texttt{FunDist}_{\texttt{Delta}}(f, g) \triangleq \texttt{Delta}(f, g)$ where f is considered as $\texttt{Graph}(f)$ and g as $\texttt{Graph}(g)$.

Proposition 7 ($\star$)**.** *Let X be a finite set, $\texttt{FunDist}_{\texttt{Delta}}$ is a distance between functions from X to any other set.*

This is a consequence of $\texttt{Delta}$ being a distance between finite sets.
Suppose we already now a distance d on X and a distance d' on Y, then we have a distance on $X \times Y$ with $\texttt{Haus}_{\texttt{Sum}(d,d')}$. So we can define distances between functions:

Definition 24 ($\star$)**.** Let $\langle X, d\rangle$ and $\langle Y, d'\rangle$ be two metric spaces, we define the functions on $Y^X \times Y^X$ such that $\forall f, g \in Y^X$: $\texttt{FunDist}_{\texttt{Haus},d,d'}(f, g) \triangleq \texttt{Haus}_{\texttt{Sum}(d,d')}(f, g)$ and $\texttt{FunDist}_{\texttt{Inj},d,d'}(f, g) \triangleq \texttt{Inj}_{\texttt{Sum}(d,d')}(f, g)$ where f is considered as $\texttt{Graph}(f)$ and g as $\texttt{Graph}(g)$.

Proposition 8 ($\star$)**.** *Let $\langle X, d\rangle$ and $\langle Y, d'\rangle$ be two metric spaces, $\texttt{FunDist}_{\texttt{Haus},d,d'}$ and $\texttt{FunDist}_{\texttt{Inj},d,d'}$ are distances between functions from X to Y.*

One could also want to compare the images of a same element through two different functions. For example, comparing $f(x)$ to $g(y)$ seems less relevant than comparing $f(x)$ and $g(x)$ when comparing f and g. The Hamming distance counts the points where two valuations differ. This is relevant because valuations can take on two values (0 or 1). Let's extend this idea to functions taking values in a metric space:

Definition 25 ($\star$)**.** Let X be a set and $\langle Y, d\rangle$ be a finite metric space, and let f and g be two functions from X to Y. We define: $\texttt{ExtHam}_d(f, g) \triangleq \sum_{x \in X} d(f(x), g(x))$.

$\texttt{Ham}$ then is exactly $\texttt{ExtHam}_{\texttt{Drast}}$ applied to valuations.

Proposition 9 ($\star$)**.** *Let X be a finite set and $\langle Y, d\rangle$ be a finite metric space, $\texttt{ExtHam}_d$ is a distance on Y^X.*

We now know how to define distances between relations on a set X and between functions from a set X to a set Y, using as starting points a distance on X and a distance on Y. Those results are very general and will be applied to Kripke semantics in Section 4.

3.1.7 Combination with sum

Adding two distances is a way to accumulate the information out of two different distances.

Definition 26 ($\star$)**.** Let X be a set and d, d' be two distances on X, let $x, y \in X$, we define:

$$\texttt{Plus}_{d,d'}(x, y) \triangleq d(x, y) + d'(x, y)$$

Proposition 10 ($\star$)**.** *Let X be a set and d, d' be two distances on X, $\texttt{Plus}_{d,d'}$ is a distance on X.*

Remark 1 : A distance multiplicated by any constant number which is not 0 is also a distance. This allows to give weight to our distances in order to give them more impact or to compensate a natural difference of scale.
If one distance is strictly more important than the other, it is possible to simulate the lexicographic order with the following distance:

Definition 27 ($\star$)**.** Let X be a set and d, d' be two distances on X, let $x, y \in X$, we define:

$$\texttt{Lex}_{d,d'}(x, y) \triangleq d(x, y) \times \texttt{Dmax}(X, d) + d'(x, y)$$

Proposition 11 ($\star$)**.** *Let X be a set and d, d' be two distances on X,* $\texttt{Lex}_{d,d'}$ *is a distance on X.*

Remark 2 : This same method can be also used with `Sum` to give more importance to some elements of the tuples.

3.2 Axiomatic study of the candidates

The axioms satisfied by the previous pre-distances are summerized in figure 6. The proofs are to be found in the appendix, with a more complete axiomatic study. On figure 6, blue cells stand for the cases where the result depends on d, d_1 or d_2. Grey cells stand for non relevant cases. "S" and asterixs stand for special cases which are specified in the proofs in appendix.

		Pre-distances		Pre-distances Between sets		Pre-distances between Subsets of a metric space				Pre-distances Between Tuples
		Drast	Plus(d1,d2)	Delta	Bin	Inj(d)	Haus(d)	PointSet(d)	Pairs(d)	Sum(d,...)
Axioms on pre-distances	CR1	V	V	V	V	V	V	V	X**	V
	CR2	V	V	V	V	V	V	V	V	V
	CR3	V	V	V	V	V	V	X	X	V
Axioms on pre-distances Between sets	CR5	V		V	X	V	X	X	X	
	CR6	V		V	X	X	X	X	X	
	CR7 str	X		X	X	X	X	X	V	
	CR8 K	X		V	X	X	X	X	X	
	CR16	X		V	X	X	X	X	X	
Axioms on pre-distances between Substes of a metric space	d-CR11					X*	V	X	X	
	d-CR17a	S****				V	V	V***	V	
Pre-distance between tuples	d-CR12									V

Figure 6: Axiomatic study of the introduced pre-distances

The properties on figure 6 formalises the fact that `Drast` is very unexpressive. We will use `Drast` only when we need to know if two elements are different or not (thanks to (CR2)) or when we have no other choice.

(CR5) and (CR6) show that `Delta` treats each element in the same way. This can be seen as a strong point if yout purpose is to count the number of elements differing betweena set A and a set B, which is exactly what `Delta` does. This can also be seen as an inconvenient because `Delta` does not consider the distance between the elements of A and B. The fact that `Delta` "counts" the elements differing between A and B is made explicit by (CR8).

`Bin` appears to be very unfitted for distance-based belief revision, even less than `Drast` based on the number of axioms they both satisfy. More concretely, `Bin` neither gives an information about cardinality, nor does it give an information about the distance of the elements of the sets. The only information given by `Bin` is about the binary numbers representing the subsets, which is not relevant information for belief-revision.

(CR11) expresses the fact that `Haus` gives a precise information about the distance between some elements in the compared subsets. (CR5) expresses the fact that `Inj` is less precise about distance between elements in the case of $A \subseteq B$.

3.3 Characterization of some candidates

Proposition 12 ($\star$)**.** *Let X be a finite set,* `Delta` *is the only pre-distance satisfying* (CR1)*,* (CR2) *and* (CR8_1) *on* $\texttt{Pow}(X)$*.*

The previous characterization teaches us that `Delta` is the only distance giving a precise information about the cardinality of the difference between two sets.

Proposition 13 ($\star$)**.** *Let K be a real number,* $K \times$ `Delta` *is the only pre-distance satisfying* (CR1)*,* (CR2) *and* (CR1_K)*.*

Proposition 14 ($\star$)**.** *Let $\langle X, d\rangle$ be a metric space, $\texttt{Pairs}_d$ is the only function on X satisfying* (CR1$_W$)*,* (CR2)*,* (CR7$_{\texttt{Str}}$) *and* (d-CR17a)*.*

3.4 For the sake of generalisation

In subsetction 3.1.4, we define several pre-distances between subsets of a metric space, using the distance of the metric space. For example, with a metric space $\langle X, d\rangle$, $\texttt{Haus}_d$ is constructed using d. We will need to combine these methods several times, so we need a more general notation, and we must get rid of indices. From now on, we will therefore also use the notation $F(d)$ as a pre-distance variable, which can be either $\texttt{Haus}_d$, $\texttt{Inj}_d$ or any pre-distance constructed on d. $\texttt{Haus}$ is then the function associating $\texttt{Haus}_d$ to d for example, and F is a function variable from distances on X to distances on $\texttt{Pow}(X)$. We then call F a *set-distance function*.

Definition 28 ($\star$)**.** A *set-predistance function* F is a function that maps to any pre-distance d on a set X a pre-distance on $\texttt{Pow}(X)$. If F is such that if d being distance implies $F(d)$ being a distance, then F is called a *set-distance function*.

It then becomes natural to write $\texttt{Haus}(d)$ for $\texttt{Haus}_d$, and so on.

Definition 29 ($\star$)**.** Let $\langle X, d\rangle$ be a metric space. We define: $\texttt{Haus}(d) \triangleq \texttt{Haus}_d$, $\texttt{Inj}(d) \triangleq \texttt{Inj}_d$, and so on.

With the same idea :

Definition 30 ($\star$)**.** A *tuple-distance function* F is a function that maps to any family of pre-distances $\mathbf{d} = (d_i)_{i\in[\![1,n]\!]}$ on a family of sets $(X_i)_{i\in[\![1,n]\!]}$ a pre-distance on $X_1 \times ... \times X_n$ and such that if d is a distance, then $F(\mathbf{d})$ is a distance.

The notation $\texttt{Sum}(\mathbf{d})$ instead of $\texttt{Sum}_{\mathbf{d}}$ also becomes natural.

Definition 31 ($\star$)**.** A *relation-predistance function* R is a function that maps a pre-distance between relations on a set X to a triple of a set-predistance S, a tuple-distance T and a pre-distance d on X.
Besides, if R is such that S being a set-distance and d being a distance implies that $R(S, T, d)$ is a distance, then R is called a *relation-distance function*.

Definition 32 ($\star$)**.** Let $\langle X, d\rangle$ be a metric space, S a set-predistance function, T a tuple-distance function, we define: $\texttt{RDF}(S, T, d) \triangleq S(T(d, d))$.

Definition 33 ($\star$)**.** A *function-predistance function* is a function F that maps a pre-distance between functions from a set X to a set Y to a quadruple of a set-predistance function S, a tuple-distance T, a pre-distance d on X and a pre-distance d' on Y.
Besides, if F is such that S being a set-distance function, d and d' being distances implies $F(S, T, d, d')$ being a distance, then F is called a *function-distance function*.

Definition 34 ($\star$)**.** Let $\langle X, d\rangle$ and $\langle Y, d\rangle$ be two metric spaces, S a set-predistance function, T a tuple-distance function, we define: $\texttt{FDF}(S, T, d, d') \triangleq S(T(d, d'))$.

Proposition 15 ($\star$)**.** $\texttt{Haus}$ *and* $\texttt{Inj}$ *are set-distance functions,* $\texttt{Sum}$ *is a tuple-distance,* $\texttt{RDF}$ *is a relation-distance function and* $\texttt{FDF}$ *is a function-distance function.*

4 Application to belief revision in modal logic

All the following propositions are direct consequences of the properties in Section 3.

Proposition 16 ($\star$). *Let $\langle \Omega, d\rangle$ be a metric space. If P_f is a pre-distance between functions from Ω to $\mathtt{Val(A)}$, F_R is a relation-predistance function, T and T' are tuple-distance functions, S and S' are set-preditsance functions then $T(S(d), F_R(S', T', d), P_f)$ is a pre-distance between finite Kripke models taking their values in Ω.*
Besides, if P_f is a ditsance and F_R is a relation-distance function and S, S' are set-distance functions then $T(S(d), F_R(S', T', d), P_f)$ is a distance between finite Kripke models taking their values in Ω.

If the considered Kripke models take their values in a set Ω without any known distance on it, then $\langle \Omega, \mathtt{Drast}\rangle$ is always a metric space.

Corollary 1 ($\star$). *Let $\langle \Omega, d\rangle$ be a metric space. If F is a pre-distance function, d' is a pre-distance between valuations, S, S', S'' are set-predistance functions, F_R is a relation-predistance function, T, T', T'' are tuple-distance functions, then $T(d, F_R(S, T', d), F(S', T'', d, d'))$ is a pre-distance between finite Kripke models taking their values in Ω.*
Besides, if F is a function-distance function, S, S', S'' are set-distance functions, d' is a distance, and F_R is a relation-distance function, then $T(S(d), F_R(S', T', d), F(S'', T'', d, d'))$ is a distance between finite Kripke models taking their values in Ω.

Corollary 2 ($\star$). *The following functions are distances between finite Kripke models:*
- $\mathtt{Sum}(\mathtt{Haus}(\mathtt{Drast}), \mathtt{RDF}(\mathtt{Haus}, \mathtt{Sum}, \mathtt{Drast}), \mathtt{FDF}(\mathtt{Haus}, \mathtt{Sum}, \mathtt{Drast}, \mathtt{Ham}))$
- $\mathtt{Sum}(\mathtt{Inj}(\mathtt{Drast}), \mathtt{RDF}(\mathtt{Inj}, \mathtt{Sum}, \mathtt{Drast}), \mathtt{FDF}(\mathtt{Inj}, \mathtt{Sum}, \mathtt{Drast}, \mathtt{Ham}))$
- $\mathtt{Sum}(\mathtt{Haus}(\mathtt{Drast}), \mathtt{RDF}(\mathtt{Haus}, \mathtt{Sum}, \mathtt{Drast}), \mathtt{ExtHam}_{\mathtt{Ham}})$
- $\mathtt{Sum}(\mathtt{Haus}(\mathtt{Drast}), \mathtt{RDF}(\mathtt{Inj}, \mathtt{Sum}, \mathtt{Drast}), \mathtt{Delta})$
- $\mathtt{Sum}(\mathtt{Delta}, \mathtt{Delta}, \mathtt{Delta})$

and so on...

If there is a distance on Ω, `Drast` can be replaced by this distance.
The nature of the elements of W in a Kripke model $\langle W, R, f\rangle$ has no effect on the interpretation of formulas. The following functions can therefore seem more relevant than the previous ones, even though they are not mathematically distances.

Definition 35 ($\star$).
$\mathtt{D}_1(\langle W, R, f\rangle, \langle W', R', f'\rangle) \triangleq \mathtt{RDF}(\mathtt{Haus}, \mathtt{Sum}, \mathtt{Drast})(R, R') + \mathtt{ExtHam}_{\mathtt{Ham}}(f, f')$
$\mathtt{D}_2(\langle W, R, f\rangle, \langle W', R', f'\rangle) \triangleq \mathtt{RDF}(\mathtt{Inj}, \mathtt{Sum}, \mathtt{Drast})(R, R') + \mathtt{ExtHam}_{\mathtt{Ham}}(f, f')$
$\mathtt{D}_3(\langle W, R, f\rangle, \langle W', R', f'\rangle) \triangleq \mathtt{RDF}(\mathtt{Haus}, \mathtt{Sum}, \mathtt{Drast})(R, R') + \mathtt{ExtHam}_{\mathtt{Ham}}(f, f')$
And so on...

The possible combinations are numerous and the relevance of each is still to be discussed and studied.

5 Conclusion and perspectives

The goal of this internship was to find a distance between Kipke models, in order to extend distance-based belief revision to modal logic. A second, optional part of the internship was to study the properties of the found distance and try to characterize it, if time allowed it. I chose to break the problem down several steps: I found one distance between subsets of a metric space: `Inj` and I studied the properties of two others `Delta` and `Haus`. I introduced and assembled a range of construction methods allowing to define distances between different kinds of mathematical objects. Those methods are general and can be applied to a lot of different domains of research, beginning with the extension of distance-based revision for other sorts of logics.

I then applied those methods to construct distances and pre-distances between Kripke models. The offered possibilities for constructing such pre-distances are numerous, this is why the axiomatic study of the distances is a main point. Next year, a PhD student will study further the subject of distance-based revision in modal logic. It is still to be determined, which axioms are desirable for the pre-distance to satisfy. Once this point will be cleared, the axiomatic study of my pre-distances will allow to determine which is the most relevant for distance-based revision.

The impossibility result on (CR4) expresses a surprising limit on what pre-distances between subsets of a metric space can express. The characterization of `Delta` shows that it is the only pre-distance giving a precise indication about the cardinality of the difference between subsets. The characterization of `Pairs` shows that ($\mathrm{CR7}_{\mathtt{Str}}$) is not a relevant axiom for a distance to satisfy.

My tutors encouraged me to write an article to submit to the journal IJAR, which is now in progress

A Proofs

A.1 Proofs of section 2

Proof. (($\texttt{Ham}$-CR4) *cannot be satisfied*)
Remark 3 : We note that we can reach a similar contradiction as in the counter-example for $\texttt{Eucl}$ on any metric space $\langle X, d\rangle$ as long as there exist $a, b, c, d, e, \in X$ such that $\begin{cases} d(a,d) > d(a,e) \\ d(b,e) > d(b,d) \end{cases}$
Using the previous remark: suppose there exists F satisfying ($\texttt{Ham}$-CR4). We find some valuations a, b, c, d, e such that:
$\begin{cases} \texttt{Ham}(a,d) > \texttt{Ham}(a,e) \\ \texttt{Ham}(b,e) > \texttt{Ham}(b,d) \end{cases}$
Fo example, if $\#X = 3$, we can take $a = (0,0,0)$, $b = (1,1,1)$, $c = (1,0,1)$, $d = (1,1,0)$ and $e = (1,0,0)$.

Then $\texttt{Ham}(a,d) = 2 > 1 = \texttt{Ham}(a,e)$ and $\texttt{Ham}(b,e) = 2 > 1 = \texttt{Ham}(b,d)$.
So we find the same contradiction again.

□

Proof. (*Relations between axioms*)

- **(CR1) + (CR2) + (CR8) ⇒ (CR3), (CR5), (CR6):** This is a direct consequence of proposition 12 in Subsection 3.1.3.

- **(CR8) ⇒ (CR6), (CR16):** Let X be a set of finite sets and F be a pre-distance on X satisfying (CR8). Let $x, y \in \bigcup X$ and $A, B \in X$ such that $\{x, y\} \cap (A \cup B) = \emptyset$.
$F(A \cup \{x\}, B) = F(A, B) + 1 = F(A \cup \{y\}, B)$ so F satisfies (CR6).
$F(A \cup \{x\}, B) = F(A\, B) + 1 > F(A, B)$ so F satisfies (CR16).

- **(CR9) ⇒ (CR16):** Let X be a set of finite sets and F be a pre-distance on X satisfying (CR9). Let $x \in \bigcup X$ and $A, B \in X$ such that $\{x\} \cap (A \cup B) = \emptyset$.
$\#((A \cup \{x\})\Delta B) > \#(A\Delta B)$ so $F((A \cup \{x\}), B) > F(A, B)$. So F satisfies (CR16).

- **(d-CR12S$_X$) ⇒ (d-CR12$_X$) (with X = L,R or n):** Direct.

- **(d-CR11) ⇒ (d-CR17a):** Direct.

- **(CR7$_{\texttt{Str}}$) ⇒ $\forall x \in X,\ F(\emptyset, \{x\}) = 0$:**
Let X be a set of finite sets and F be a pre-distance on X satisfying (CR7$_{\texttt{Str}}$), let $x, y \in \bigcup X \mid x \neq y$, we have $F(\{y\}, \{x\}) = F(\emptyset, \{x\}) + F(\{y\}, \{x\})$ so $F(\emptyset, \{x\}) = 0$

- **(CR6) + (CR16) ⇏ (CR8):** Let X be a set of finite sets, let $A, B \in X$, we define:
$\texttt{F}(A, B) \triangleq (\#A)!$
$\texttt{F}$ satisfies (CR6) and (CR16) and does not satisfy (CR8).

- **(CR2) + (CR8) ⇏ (CR1):** Let X be a set of finite sets, let $A, B \in X$, we define:
$\texttt{F}(A, B) \triangleq \#A + \#B$
$\texttt{F}$ satisfies (CR2) and (CR8) and does not satisfy (CR1).

- **(d-CR12$_X$) ⇏ (d-CR12S$_X$) (with X = L,R or n):** Let $\texttt{X}_1 \triangleq \{1,2,3\}$ and $\texttt{X}_2 \triangleq \{4,5,6\}$. Let $x, y \in \mathbb{N}$, we define $\texttt{d}(x,y) = |x - y|$ ($\texttt{d}$ is a distance on X_1 and on X_2). Then let $\langle x, y\rangle, \langle x', y'\rangle \in X_1 \times X_2$. We define $\texttt{F}(\langle x, y\rangle, \langle x', y'\rangle) \triangleq d(x, x') + d(y, y')$.
$\texttt{F}$ satisfies ($\texttt{d}$-CR12$_1$) but not ($\texttt{d}$-CR12S$_1$). Indeed:
Let $x, x', y, y' \in X_1$, $z, z' \in X_2$ such that $\texttt{d}(x, x') < \texttt{d}(y, y')$. Then $\texttt{F}(\langle x, z\rangle, \langle x', z'\rangle) = d(x, x') + d(z, z') < d(y, y') + d(z, z') = \texttt{F}(\langle y, z\rangle, \langle y', z'\rangle)$ so $\texttt{F}$ satisfies ($\texttt{d}$-CR12$_1$).
But $\texttt{d}(1,1) = 0 < 2 = \texttt{d}(1,3)$ and still:
$\texttt{F}(\langle 1, 0\rangle, \langle 1, 3\rangle) = 3 > 2 = \texttt{F}(\langle 1, 3\rangle, \langle 3, 3\rangle)$. So $\texttt{F}$ does not satisfy ($\texttt{d}$-CR12S$_1$).

- **(d-CR17a) ⇏ (d-CR11):** $\texttt{Pairs}$ satisfies (d-CR17a) but not (d-CR11) (see Subsection **??**).

- **(CR16) ⇏ (CR9):** Let X be a set of finite sets, let $A, B \in X$, we define:
$\texttt{F}(A, B) \triangleq \#A$
$\texttt{F}$ satisfies (CR16) but not (CR9). Indeed :
With $X = \{1, 2, 3\}$:
$F(\{1,2,3\}, \{1,2,3\}) = 3 > 0 = F(\emptyset, \{1\})$ and $\#(\{1,2,3\}\Delta\{1,2,3\}) = 0 < 1 = \#(\emptyset\Delta\{1\})$

- **Let F be a function, F can satisfy at the most one strong monotony: (d_1-CR12S$_{\text{L}}$), (d_2-CR12S$_{\text{R}}$) or (d_n-CR12S$_n$) with an unique n:** Let $N \in \mathbb{N}$ and X a set of N-tuples. X can be expressed as $X_1 \times ... \times X_N$. Let $n, m \in [\![1, N]\!]$ such as $n \neq m$. Let d_n be a distance on X_n and d_m a distance on X_m. Suppose F satisfies (d_n-CR12S$_n$) and (d_m-CR12S$_m$).
Let $a, a', b, b' \in X_n$ such that $d_n(a, a') < d_n(b, b')$ and x, x', y, y' such that $d_m(x, x') > d_m(y, y')$. Let $u, u', v, v' \in X$ such that $u_n = a$ (ie. the n-th term of u is a), $u_m = x$, $u'_n = a'$, $u'_m = x'$, $v_n = b$, $v_m = y$, $v'_n = b'$, $v'_m = y'$.
Then $F(u, u') < F(v, v')$ and $F(u, u') > F(v, v')$. Contradiction.

- **(CR7$_{\texttt{Str}}$) $\Rightarrow$ $\neg$(CR1), $\neg$(d-CR17b):** Let X be a set of finite sets and F be a pre-distance on X satisfying (CR7$_{\texttt{Str}}$). Let $x \in \bigcup X$, let d be a distance on $\bigcup X$ and suppose that $\# \bigcup X \geqslant 2$, then $\texttt{Dmax}(\bigcup X, d) > 0$. $F(\emptyset, \{x\}) = 0$ (see above)
So $F(\emptyset, \{x\}) \neq \texttt{Dmax}(\bigcup X, d)$ and $F(\emptyset, \{x\}) = 0$ even though $\emptyset \neq \{x\}$. So F does not satisfy (d-CR17b) nor (CR1).

- **(CR1) + (CR2) + (CR8) $\Rightarrow$ $\neg$(CR7$_{\texttt{Far}}$), $\neg$(CR7$_{\texttt{Str}}$):** This is a direct consequence of proposition 12 in Subsection 3.1.3.

- **(d_1-CR12$_{\text{L}}$) and (d_2-CR12$_{\text{R}}$) are compatible. As well as (d_n-CR12$_n$) and (d_m-CR12$_m$) with $n \neq m$:** Let $N \in \mathbb{N}$ and $(\langle X_i, d_i \rangle)_{i \in [\![1,N]\!]}$, let $\mathbf{b} \triangleq (d_i)_{i \in [\![1,N]\!]}$, then $\texttt{Sum}_{\mathbf{d}}$ (defined in lemma 1) satisfies (d_i-CR12$_i$) for all $i \in [\![1, N]\!]$.

- **(CR1), (CR2), (CR3), (CR5), (CR6), (CR8), (CR9), (CR16) and (CR18) are compatible:** `Delta` satifies all of these axioms (see Subsection 3.1.3).

- **(CR1), (CR2), (CR3), (CR11), (CR17a) and (CR17b) are compatible:** `Haus` satifies all of these axioms (see Subsection **??**).

- **(CR1), (CR2), (CR3), (CR5), (CR17a) and (CR17b) are compatible:** `Inj` satifies all of these axioms (see Subsection **??**).

□

A.2 Proofs of section 3

Proof. (*Lemma 1*)

We show that $\texttt{Sum}_{\mathbf{d}}$ satisfies the three properties of distances:

Let $n \in \mathbb{N}$, $(\langle X_i, d_i \rangle)_{i \in [\![1,n]\!]}$ be n metric spaces. Let $x, y, z \in \mathbf{X}$.

- $\texttt{Sum}_{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = 0 \iff \forall i \in [\![1, n]\!],\, d(x_i, y_i) = 0 \iff \forall i \in [\![1, n]\!],\, x_i = y_i \iff \mathbf{x} = \mathbf{y}$
- $\texttt{Sum}_{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} d_i(x_i, y_i) = \sum_{i=1}^{n} d_i(y_i, x_i) = \texttt{Sum}_{\mathbf{d}}(\mathbf{y}, \mathbf{x})$
- $\forall i \in [\![1, n]\!]$, $d(x_i, y_i) \leqslant d(x_i, z_i) + d(z_i, y_i)$ so:
$\texttt{Sum}_{\mathbf{d}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} d_i(x_i, y_i) \leqslant \sum_{i=1}^{n} d_i(x_i, z_i) + \sum_{i=1}^{n} d_i(z_i, y_i) = \texttt{Sum}_{\mathbf{d}}(\mathbf{x}, \mathbf{z}) + \texttt{Sum}_{\mathbf{d}}(\mathbf{z}, \mathbf{y})$

□

Proof of `Delta` being a distance:

Lemma 2 ($\star$)**.** *For all sets A, B, C, we have $A \Delta B \subseteq A \Delta C \cup C \Delta B$.*

Proof. (*Lemma 2*)
Let A, B, C be three sets.

Let $x \in A \Delta B$, then we have:

- Either $x \in A \cap B^c$:
  - If $x \in C$: $x \in C \cap B^c \subseteq A\Delta C \cup C\Delta B$.
  - Else $x \in C^c$: $x \in A \cap C^c \subseteq A\Delta C \cup C\Delta B$.
- Either $x \in B \cap A^c$:
  - If $x \in C$: $x \in C \cap A^c \subseteq A\Delta C \cup C\Delta B$.
  - Else $x \in C^c$: $x \in B \cap C^c \subseteq A\Delta C \cup C\Delta B$.

Eitherways: $x \in A\Delta C \cup C\Delta B$ so $A\Delta B \subseteq A\Delta C \cup C\Delta B$. □

Proof. (`Delta` *is a distance*) We show that `Delta` satisfies the three properties of distances:

- If $A = B$ then $A \cap B^c = B \cap A^c = \emptyset$ so $A\Delta B = \emptyset$ and $\#(A\Delta B) = 0$.
  Reciprocally, if $\#(A\Delta B) = 0$ then $A\Delta B = \emptyset$ so $A \cap B^c = B \cap A^c = \emptyset$ so $A \subseteq B$ and $B \subseteq A$ so $A = B$.
- By symetry of the intersection and then union operators, $A\Delta B = B\Delta A$ so $\#(A\Delta B) = \#(B\Delta A)$.
- By the previous lemma $A\Delta B \subseteq A\Delta C \cup C\Delta B$. So:
  $\#(A\Delta B) \leqslant \#(A\Delta C \cup C\Delta B) \leqslant \#(A\Delta C) + \#(C\Delta B)$

□

Proof. (`Bin` *is a distance*) Let X be a finite set, we order its elements: $\bigcup X = \{x_1, x_2, ...\}$. Let $A, B, C \subseteq X$.

- $\texttt{Bin}(A, B) = 0 \iff \texttt{Numb}(A) = \texttt{Numb}(B)$. Since the correspondance between $\mathbb{1}_S$ and $\texttt{Numb}(S)$ for any subset S of X is a bijection (each binary number corresponds to an unique element of $\mathbb{N}$): $\texttt{Numb}(A) = \texttt{Numb}(B) \iff \mathbb{1}_A = \mathbb{1}_B \iff A = B$.
- $\texttt{Bin}(A, B) = |\texttt{Numb}(A) - \texttt{Numb}(B)| = |\texttt{Numb}(B) - \texttt{Numb}(A)| = \texttt{Bin}(B, A)$.
- $\texttt{Bin}(A, B) = |\texttt{Numb}(A) - \texttt{Numb}(B)| = |\texttt{Numb}(A) - \texttt{Numb}(C) + \texttt{Numb}(C) - \texttt{Numb}(B)| \leqslant |\texttt{Numb}(A) - \texttt{Numb}(C)| + |\texttt{Numb}(C) - \texttt{Numb}(B)| = \texttt{Bin}(A, C) + \texttt{Bin}(C, B)$.

□

Proof of `Inj` being a distance:
For the proof we will need two lemmas:

Lemma 3 ($\star$)**.** *Let $\langle X, d\rangle$ be a metric space and $A, B \subseteq X \mid 0 < |A| \leqslant |B|$ eand $f : A \to B$, then:*

$$\texttt{Dist}(d, A, B, f) = 0 \iff f_{|A} = \texttt{Id}$$

Proof. (*Lemma 3*)
If $f_{|a} = \texttt{Id}$, then $\texttt{Dist}(d, A, B, f) = \texttt{Dist}(d, A, B, \texttt{Id}) = \sum_{a\in A} d(a, a) = 0$

Reciprocally, if $\texttt{Dist}(d, A, B, f) = 0$: Suppose $\exists a \in A \mid f(a) \neq a$ then $d(a, f(a)) > 0$ so $\texttt{Dist}(d, A, B, f) > 0$. Contradtiction. So $\forall a \in A, f(a) = a$ ie. $f_{|a} = \texttt{Id}$.

□

Proof. (`Inj` *is identity respecting.*)

Let $\langle X, d\rangle$ be a metric space and $A, B \subseteq X$:

Suppose $A = B$, then `Id` is an injective function from A to B and from B to A and by lemma 3:

$$\texttt{Dist}(d, A, B, \texttt{Id}) = 0 \text{ and } \texttt{Dist}(d, B, A, \texttt{Id}) = 0$$

So $\texttt{Inj}(A, B) = \min\limits_{f:A\hookrightarrow B} \texttt{Dist}(d, A, B, f) + \min\limits_{f:B\hookrightarrow A} \texttt{Dist}(d, B, A, f) + ||A| - |B|| = 0$

Reciprocally, suppose $\texttt{Inj}(A, B) = 0$, then:

$$\begin{cases} ||A|-|B|| = 0 \\ \min\limits_{f:A\hookrightarrow B} \mathtt{Dist}(d,A,B,f) = 0 \\ \min_{f:B\hookrightarrow A} \mathtt{Dist}(d,B,A,f) = 0 \end{cases}$$

So $|A| = |B|$, which implies $\{f : A \hookrightarrow B\} \neq \emptyset$ and $\{f : B \hookrightarrow A\} \neq \emptyset$, then:
$\exists f_1 \in \{f : A \hookrightarrow B\} \mid \mathtt{Dist}(d,A,B,f_1) = 0$ ie. $\mathtt{Id} \in \{f : A \hookrightarrow B\}$ (by lemma 3). ie. $A \subseteq B$.
The same way: $\exists f_2 \in \{f : B \hookrightarrow A\} \mid \mathtt{Dist}(d,B,A,f_2) = 0$ ie. $\mathtt{Id} \in \{f : B \hookrightarrow A\}$ ie. $B \subseteq A$.
So $A = B$.

□

Proof. ($\mathtt{Inj}$ *is symmetric.*)

$$\begin{aligned} \mathtt{Inj}(A,B) &= \min_{f:A\hookrightarrow B} \mathtt{Dist}(d,A,B,f) + \min_{f:B\hookrightarrow A} \mathtt{Dist}(d,B,A,f) + ||A|-|B|| \\ &= = \min_{f:B\hookrightarrow A} \mathtt{Dist}(d,B,A,f) + \min_{f:A\hookrightarrow B} \mathtt{Dist}(d,A,B,f) + ||B|-|A|| \\ &= \mathtt{Inj}(B,A) \end{aligned}$$

□

Lemma 4 ($\star$). *Let $\langle X,d\rangle$ be a metric space and $A,B,C \subseteq X$, $\varphi : A \hookrightarrow C, \psi : C \to B$, then:*

$$\mathtt{Dist}(d,A,B,\psi\circ\varphi) \leqslant \mathtt{Dist}(d,A,C,\varphi) + \mathtt{Dist}(d,C,B,\psi)$$

Proof. (*Lemma 4*)

Let $\langle X,d\rangle$ be a metric space and $A,B,C \subseteq X$, $\varphi : A \hookrightarrow C, \psi : C \to B$.

$$\begin{aligned} \mathtt{Dist}(d,A,B,\psi\circ\varphi) &= \sum_{a\in A} d(a,\psi\circ\varphi(a)) \\ &\leqslant \sum_{a\in A} (d(a,\varphi(a)) + d(\varphi(a),\psi\circ\varphi(a))) \\ &= \sum_{a\in A} d(a,\varphi(a)) + \sum_{c\in\varphi(A)} d(c,\psi(c)) \qquad \text{because } \varphi \text{ is an injective function} \\ &\leqslant \sum_{a\in A} d(a,\varphi(a)) + \sum_{c\in C} d(c,\psi(c)) \\ &= \mathtt{Dist}(d,A,C,\varphi) + \mathtt{Dist}(d,C,B,\psi) \end{aligned}$$

□

Proof. ($\mathtt{Inj}$ *satisfies the triangle inequality.*)

We note that:

$$(*) \begin{cases} \{\psi\circ\varphi \mid \varphi : A \hookrightarrow C, \psi : C \hookrightarrow B\} \subseteq \{f : A \hookrightarrow B\} \\ \{\psi\circ\varphi \mid \varphi : B \hookrightarrow C, \psi : C \hookrightarrow A\} \subseteq \{f : B \hookrightarrow A\} \end{cases}$$

Then:

$$\begin{aligned} \mathtt{Inj}(A,C) + \mathtt{Inj}(C,B) &= \min_{\varphi:A\hookrightarrow C} \mathtt{Dist}(d,A,C,\varphi) + \min_{\varphi:C\hookrightarrow A} \mathtt{Dist}(d,C,A,\varphi) + ||A|-|C|| \\ &\quad + \min_{\psi:C\hookrightarrow B} \mathtt{Dist}(d,C,B,\psi) + \min_{\psi:B\hookrightarrow C} \mathtt{Dist}(d,B,C,\psi) + ||C|-|B|| \\ &\geqslant ||A|-|B|| + \min_{\substack{\varphi\,:\,A\,\hookrightarrow\,C \\ \psi\,:\,C\,\hookrightarrow\,B}} (\mathtt{Dist}(d,A,C,\varphi) + \mathtt{Dist}(d,C,B,\psi)) \\ &\quad + \min_{\substack{\psi\,:\,B\,\hookrightarrow\,C \\ \varphi\,:\,C\,\hookrightarrow\,A}} (\mathtt{Dist}(d,B,C,\psi) + \mathtt{Dist}(d,C,A,\varphi)) \\ &\geqslant ||A|-|B|| + \min_{\substack{\varphi\,:\,A\,\hookrightarrow\,C \\ \psi\,:\,C\,\hookrightarrow\,B}} \mathtt{Dist}(d,A,B,\psi\circ\varphi) \\ &\quad + \min_{\substack{\psi\,:\,B\,\hookrightarrow\,C \\ \varphi\,:\,C\,\hookrightarrow\,A}} \mathtt{Dist}(d,B,A,\psi\circ\varphi) \qquad \text{(by lemma 4)} \\ &\geqslant ||A|-|B|| + \min_{f:A\hookrightarrow B} \mathtt{Dist}(d,A,B,f) + \min_{\psi:B\hookrightarrow A} \mathtt{Dist}(d,B,A,f) \qquad \text{(by } (*)\text{)} \\ &= \mathtt{Inj}(A,B) \end{aligned}$$

□

Proof. ($\texttt{PointSet}_d$ *is identity respecting and symmetric but does not satisfy the triangle inequality*)

Let $\langle X, d\rangle$ be a metric space, $A, B, C \in X$:

- Suppose $A = B$:

  Then $\forall a \in A, a \in B$ so $\forall a \in A, \texttt{dist}(a, B, d) = \min\{d(a, b) | b \in B\} = d(a, a) = 0$ as well as $\forall b \in B, b \in A$ gives $\forall b \in B, \texttt{dist}(b, A, d) = 0$. So $\texttt{PointSet}_d(A, B) = \sum_{a\in A} 0 + \sum_{b\in B} 0 = 0$.

  Reciprocally, suppose $\texttt{PointSet}_d(A, B) = 0$. Then:

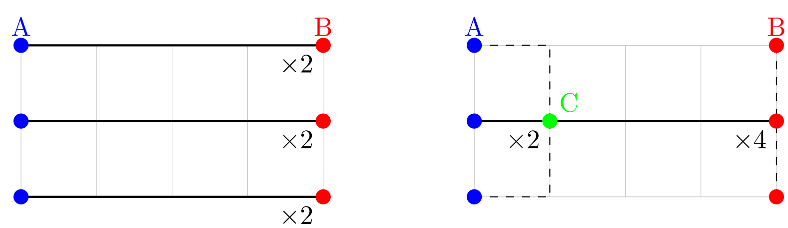
  - either both sums are empty, then $A = \emptyset = B$ so $A = B$.
  - or one is empty and the other has only null terms. Then (*wlog*) $A = \emptyset$ and $\forall b \in B, \texttt{dist}(b, A, d) = 0$, but $\texttt{dist}(b, A, d) = \min(\emptyset) = \texttt{Dmax}(X, d) > 0$. This case is impossible.
  - or both sums are non-empty and have only null terms. Then $\forall b \in B, \texttt{dist}(b, A, d) = 0$ so $\forall b \in B, \exists a \in A \mid d(a, b) = 0$ *ie.* $a = b$ *ie.* $b \in A$ so $B \subseteq A$. The same way: $\forall a \in A, \texttt{dist}(a, B, d) = 0$ gives us $A \subseteq B$ so $A = B$.

  So $\texttt{PointSet}_d$ respects identity.

- Symmetry:

$$\begin{aligned}
\texttt{PointSet}_d(A, B) &= \sum_{a\in A} \texttt{dist}(a, B, d) + \sum_{b\in WB} \texttt{dist}(b, A, d)\\
&= \sum_{b\in B} \texttt{dist}(b, A, d) + \sum_{a\in A} \texttt{dist}(a, B, d)\\
&= \texttt{PointSet}_d(B, A)
\end{aligned}$$

- In order to prove that $\texttt{PointSet}_d$ does not respect the triangle inequality, we use the following counter-example, with X being a set of point of the space and d being the Manathan distance $\texttt{Man}$.



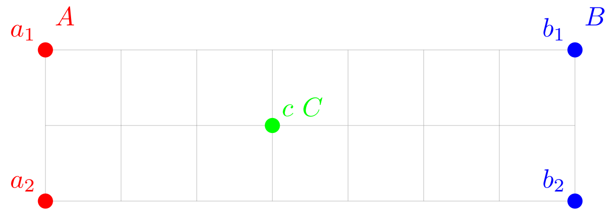
  $\texttt{PointSet}_{\texttt{Man}}(A, B) = 4 \times 2 \times 3 = 24$
  $\texttt{PointSet}_{\texttt{Man}}(A, C) + \texttt{PointSet}_{\texttt{Man}}(C, B) = 2 + 2 + 1 \times 2 + 3 \times 4 + 1 + 1 = 20$

  So we have $\texttt{PointSet}_{\texttt{Man}}(A, B) > \texttt{PointSet}_{\texttt{Man}}(A, C) + \texttt{PointSet}_{\texttt{Man}}(C, B)$. $\texttt{PointSet}_{\texttt{Man}}$ does not respect the triangle inequality.

□

Proof. ($\texttt{Pairs}$ *is symmetric but it is neither identity respecting, nor does it respect the triangle inequality.*)

- The symmetry was proved in the proof of proposition 14.
- Let $\langle X, d\rangle$ be a metric space and $z \in X$. $\texttt{Pairs}_d(\{z\}, \emptyset) = \sum_{(x,y)\in\{z\}\times\emptyset} d(x, y) - \sum_{(x,y)\in(\{z\}\cap\emptyset)^2} d(x, y) = 0$ and $\{z\} \neq \emptyset$.
- For the triangle inequality, we use the following counter-example, with X being a set of points in the space and d being $\texttt{Man}$:



$\texttt{Pairs}_{\texttt{Man}}(A,B) = \texttt{Man}(a_1,b_1) + \texttt{Man}(a_1,b_2) + \texttt{Man}(a_2,b_1) + \texttt{Man}(a_2,b_2) = 7+9+9+7 = 32$
$\texttt{Pairs}_{\texttt{Man}}(A,C) + \texttt{Pairs}_{\texttt{Man}}(C,B) = \texttt{Man}(a_1,c) + \texttt{Man}(a_2,c) + \texttt{Man}(c,b_1) + \texttt{Man}(c,b_2) = 4+4+5+5 = 18$
So we have a case where $\texttt{Pairs}_{\texttt{Man}}(A,B) > \texttt{Pairs}_{\texttt{Man}}(A,C) + \texttt{Pairs}_{\texttt{Man}}(C,B)$.

$\square$

Proof. ($\texttt{ExtHam}$ *is a distance.*)
Let X be a finite set set and $\langle Y, d\rangle$ be a finite metric space. Let f,g,h be functions from X to Y.

- If $f = g$: $\texttt{ExtHam}_d(f,g) = \sum_{x\in X} d(f(x),g(x)) = \sum_{x\in X} d(f(x),f(x)) = 0$.
  If $\texttt{ExtHam}_d(f,g) = 0$ then $\forall x \in X, d(f(x),g(x)) = 0$, ie. $\forall x \in X, f(x) = g(x)$, ie. $f = g$.
- $\texttt{ExtHam}_d(f,g) = \sum_{x\in X} d(f(x),g(x)) = \sum_{x\in X} d(g(x),f(x)) = \texttt{ExtHam}_d(g,f)$.
- $\texttt{ExtHam}_d(f,g) = \sum_{x\in X} d(f(x),g(x)) \leqslant \sum_{x\in X} (d(f(x),h(x)) + d(h(x),g(x)) = \texttt{ExtHam}_d(f,h) + \texttt{ExtHam}_d(h,g)$.

$\square$

Proof. ($\texttt{Plus}$ *is a distance.*)
Let X be a set and d, d' be two distances on X. Let $x,y,z \in X$

- Since d, d' are positive, $\texttt{Plus}_{d,d'}(x,y) = 0 \iff d(x,y) = 0$ and $d'(x,y) = 0 \iff x = y$.
- $\texttt{Plus}_{d,d'}(x,y) = d(x,y) + d'(x,y) = d(y,x) + d'(y,x) = \texttt{Plus}_{d,d'}(y,x)$.
- $d(x,y) \leqslant d(x,z) + d(z,y)$ and $d'(x,y) \leqslant d'(x,z) + d'(z,y)$ so $\texttt{Plus}_{d,d'}(x,y) = d(x,y) + d'(x,y) \leqslant d(x,z) + d(z,y) + d'(x,z) + d'(z,y) = \texttt{Plus}_{d,d'}(x,z) + \texttt{Plus}_{d,d'}(z,y)$.

$\square$

Proof. ($\texttt{Lex}$ *is a distance.*)
Thanks to the Remark 1, $\texttt{Dmax}(X,d) \times d$ is a distance. Then by proposition 10, since $\texttt{Lex}_{d,d'} = \texttt{Plus}_{\texttt{Dmax}(X,d)\times d, d'}$, it is a distance. $\square$

Proof. (*Proposition 12*
Let X be a finite set. We already know that $\texttt{Delta}$ satisfies (CR1), (CR2) and (CR8) on $\texttt{Pow}(X)$. Let's show that if a function F satisfies (CR1), (CR2) and (CR8) on $\texttt{Pow}(X)$, then $F = \texttt{Delta}$.

Let $F : \texttt{Pow}(X) \times \texttt{Pow}(X) \to [0,\infty)$ be a function satisfying (CR1), (CR2) and (CR8) on $\texttt{Pow}(X)$.

Let $A, B \subseteq X$. Since X is finite, A and B are also finite, so we can proceed by dobble induction on $\#A$ and $\#B$:

- If $\#A = 0$, then $A = \emptyset$ and:
  - If $\#B = 0$, then $B = \emptyset$ and $A = \emptyset = B$ so, since F is identity respecting, $F(A,B) = 0 = \#(A\Delta B)$.
  - Let $m \in \mathbb{N}$, such that $\forall B \in \texttt{Pow}(X), |B| = m \Rightarrow F(A,B) = |A\Delta B|$:
    If $|B| = m+1$, then $\exists B', x \mid B = B' \uplus \{x\}$ and $\#B' = m$,
    so $F(A,B) = F(B,A) = F(B' \uplus \{x\}, A)$ (F is symmetric).
    and $x \notin B'$, $x \notin \emptyset = A$ so $\{x\} \cap (A \cup B) = \emptyset$ and then, since F satisfies (CR8) :
    $F(A,B) = F(B' \uplus \{x\}, A) = F(B', A) + 1 = F(A, B') + 1 = \#(A\Delta B') + 1$.

Then:

$$\begin{aligned} A\Delta B &= (A\setminus B)\cup(B\setminus A)\\ &= (\emptyset\setminus B)\cup(B\setminus\emptyset)\\ &= \emptyset\cup B\\ &= B\\ &= B'\uplus\{x\}\end{aligned}
\qquad\Bigg|\qquad
\begin{aligned} A\Delta B' &= (A\setminus B')\cup(B'\setminus A)\\ &= (\emptyset\setminus B')\cup(B'\setminus\emptyset)\\ &= \emptyset\cup B'\\ &= B'\end{aligned}$$

So $\#(A\Delta B) = \#(A\Delta B') + 1$.
So $F(A, B) = \#(A\Delta B)$.

So $\forall m \in \mathbb{N}, \forall B \in \texttt{Pow}(X), \#B = m \Rightarrow F(\emptyset, B) = |\emptyset\Delta B|$,
ie. $\forall B \in \texttt{Pow}(\texttt{Val}(\mathcal{A})), F(\emptyset, B) = \#(\emptyset\Delta B)$.

- Let $m \in \mathbb{N}$ such that $\forall(A, B) \in (\texttt{Pow}(\texttt{Val}(X))^2, \#A = n \Rightarrow F(A, B) = \#(A\Delta B)$:
  If $\#A = n + 1$ then $\exists A', x \mid A = A' \uplus \{x\}$ and $F(A, B) = F(A' \uplus \{x\}, B)$.
  - If $x \notin B$, then $F(A, B) = F(A', B) + 1 = |A'\Delta B| + 1$ and (since $x \notin A'$):
    $$\begin{aligned} A\Delta B &= (A\setminus B)\cup(B\setminus A)\\ &= (A'\setminus B)\cup(B\setminus A')\cup\{x\}\\ &= (A'\Delta B)\uplus\{x\}\end{aligned}$$
    So $\#(A\Delta B) = \#A'\Delta B + 1$.
    So $F(A, B) = \#(A\Delta B)$.
  - If $x \in B$, then $\exists B' \mid B = B' \uplus \{x\}$ et $F(A, B) = F(A' \uplus \{x\}, B' \uplus \{x\})$.
    We define $C = (A' \cap B') \cup \{x\}$. Since F is identity respecting, $F(C, C) = 0$.
    Then we can add one by one the elements of $(A' \uplus \{x\}) \setminus C$ to C:
    $$\begin{aligned} F(A'\uplus\{x\}, C) &= F(C, C) + \#((A'\uplus\{x\})\setminus C)\\ &= 0 + \#(A'\setminus B')\end{aligned}$$
    (by a direct induction applying (CR8))
    The same way, by adding the elements of $(B' \uplus \{x\}) \setminus C$ to C:
    $$\begin{aligned} F(A'\uplus\{x\}, B'\uplus\{x\}) &= F(B'\uplus\{x\}, A'\uplus\{x\})\\ &= F(A'\uplus\{x\}, C) + \#((B'\uplus\{x\})\setminus C) \quad (1)\\ &= \#(A'\setminus B') + \#(B'\setminus A')\\ &= \#(A'\Delta B') \quad (2)\\ &= \#(A\Delta B)\end{aligned}$$
    (1): by a direct induction applying (CR8)
    (2): Because $(A' \setminus B') \cap (B' \setminus A') = \emptyset$
    So $F(A, B) = \#(A\Delta B)$.

So $\forall n \in \mathbb{N}, \forall(A, B) \in (\texttt{Pow}(\texttt{Val}(\mathcal{A})))^2, |A| = n \Rightarrow F(A, B) = \#(A\Delta B)$.
ie. $\forall(A, B) \in (\texttt{Pow}(\texttt{Val}(\mathcal{A})))^2, F(A, B) = \#(A\Delta B)$.

So if F satisfies (CR1), (CR2) and (CR8), F is necessarily then cardinal of the symmetric difference `Delta`.

□

Proof. (*Proposition 13*)
The exact same proof as when K was equal to 1 is possible, adding K instead of 1 at each stepp. □

Proof. (*Proposition 14.*)
Let $\langle X, d\rangle$ be a metric space, let F be a function from $\texttt{Pow}(X)$ to $\mathbb{R}$. Suppose that F satisfies (CR1$_W$), (CR2), (CR7$_{\texttt{Str}}$) and (d-CR17a). Then, let $A, B \in \texttt{Pow}(X)$:

$$
\begin{aligned}
F(A,B) &= F(A\cap B,B)+\sum_{x\in A\setminus B}F(\{x\},B) &(1)\\
&= F(A\cap B,A\cap B)+\sum_{x\in B\setminus A}F(\{x\},A\cap B)+\sum_{x\in A\setminus B}F(\{x\},B) &(2)\\
&= 0+\sum_{x\in B\setminus A}F(A\cap B,\{x\})+\sum_{x\in A\setminus B}F(B,\{x\}) &(3)\\
&= \sum_{x\in B\setminus A}\Big(F(\emptyset,\{x\})+\sum_{y\in A\cap B}F(\{y\},\{x\})\Big)+\sum_{x\in A\setminus B}\Big(F(\emptyset,\{x\})+\sum_{y\in B}F(\{y\},\{x\})\Big) &(4)\\
&= \sum_{\substack{x\,\in\,B\,\setminus\,A\\ y\,\in\,A\,\cap\,B}}F(\{y\},\{x\})+\sum_{\substack{x\,\in\,A\,\setminus\,B\\ y\,\in\,B}}F(\{y\},\{x\}) &(5)\\
&= \sum_{\substack{x\,\in\,B\,\setminus\,A\\ y\,\in\,A\,\cap\,B}}d(x,y)+\sum_{\substack{x\,\in\,A\,\setminus\,B\\ y\,\in\,B}}d(x,y) &(6)\\
&= \sum_{\substack{x\,\in\,B\,\setminus\,A\\ y\,\in\,A\,\cap\,B}}d(x,y)+\sum_{\substack{x\,\in\,A\,\setminus\,B\\ y\,\in\,A\,\cap\,B}}d(x,y)+\sum_{\substack{x\,\in\,A\,\setminus\,B\\ y\,\in\,B\,\setminus\,A}}d(x,y)\\
&= \sum_{(x,y)\in A\times B}d(x,y)-\sum_{(x,y)\in(A\cap B)^2}d(x,y)\\
&= \mathtt{Pairs}_d(A,B)
\end{aligned}
$$

(1): By induction, using ($\mathrm{CR7}_{\mathtt{Far}}$) on each element of $A\setminus B$.
(2): By induction, using ($\mathrm{CR7}_{\mathtt{Far}}$) on each element of $B\setminus A$.
(3): By ($\mathrm{CR1}_W$) and (CR2).
(4): By induction, using ($\mathrm{CR7}_{\mathtt{Str}}$) on each element of $A\cap B$, which is possible because $\forall x\in B\setminus A, x\notin A\cap B$, and on each element of B, which is possible because $\forall x\in A\setminus B, x\notin B$.
(5): $\forall x\in X, F(\emptyset,\{x\})=0$ (proposition 1) (6): By (CR17a).
So any function satisfying ($\mathrm{CR1}_W$), (CR2), ($\mathrm{CR7}_{\mathtt{Str}}$) and ($d$-CR17a) ond $\langle X,d\rangle$ has to be $\mathtt{Pairs}_d$. But does $\mathtt{Pairs}_d$ satisfy all those axioms ?

Let $A,B\in\mathtt{Pow}(X), z\in X\setminus(A\cup B)$:

- ($\mathrm{CR1}_W$) :
  $F(A,A)=\sum_{(x,y)\in A\times A}d(x,y)-\sum_{(x,y)\in(A\cap A)^2}d(x,y)=\sum_{(x,y)\in A\times A}d(x,y)-\sum_{(x,y)\in A\times A}d(x,y)=0$
- (CR2) :
$$
\begin{aligned}
\mathtt{Pairs}_d(A,B) &= \sum_{(x,y)A\times B}d(x,y)-\sum_{(x,y)\in(A\cap B)^2}d(x,y)\\
&= \sum_{(x,y)\in A\times B}d(y,x)-\sum_{(x,y)\in(B\cap A)^2}d(x,y)\\
&= \sum_{(x,y)\in B\times A}d(x,y)-\sum_{(x,y)\in(B\cap A)^2}d(x,y)\\
&= \mathtt{Pairs}_d(B,A)
\end{aligned}
$$
- ($\mathrm{CR7}_{\mathtt{Str}}$) :
$$
\begin{aligned}
\mathtt{Pairs}_d(A\cup\{z\},B) &= \sum_{(x,y)(A\cup\{z\})\times B}d(x,y)-\sum_{(x,y)\in((A\cup\{z\})\cap B)^2}d(x,y)\\
&= \sum_{(x,y)A\times B}d(x,y)+\sum_{(x,y)\{z\}\times B}d(x,y)-\sum_{(x,y)\in A\cap B)^2}d(x,y) &(1)\\
&= \mathtt{Pairs}_d(A,B)+\sum_{(x,y)\{z\}\times B}d(x,y)-\sum_{(x,y)\in(\{z\}\cap B)^2}d(x,y) &(2)\\
&= \mathtt{Pairs}_d(A,B)+\mathtt{Pairs}_d(\{z\},B)
\end{aligned}
$$

  (1): $z\notin A\cup B$, so $(A\cup\{z\})\cap B=A\cap B$.
  (2): $\{z\}\cap B=\emptyset$, so $\sum_{(x,y)\in(\{z\}\cap B)^2}d(x,y)=0$.

- (d-CR17a) : Let $a, b \in X$:
  If $a = b$:
  $\texttt{Pairs}_d(\{a\},\{b\}) = \sum_{(x,y)\in\{a\}\times\{b\}} d(x,y) - \sum_{(x,y)\in(\{a\}\cap\{b\})^2} d(x,y) = d(a,b) - d(a,b) = 0 = d(a,b)$
  If $a \neq b$:
  $\texttt{Pairs}_d(\{a\},\{b\}) = d(a,b) - 0 = d(a,b)$

$\texttt{Pairs}_d$ satisfies the four axioms.

$\square$

B Other results

B.1 Other axioms

B.1.1 Axioms on pre-distances

Definition 36 ($\star$)**.** Let X be a set and F be a pre-distance on X. We define:
Let D be a pre-distance on X,
(D-CR9) : (*D-monotony*) $\forall A, B \in X,\ D(A, B) < D(A', B') \Rightarrow F(A, B) < F(A', B')$

(D-CR9) seems meaningless by itself, but once a specific distance D is defined on X, with interesting monotony properties, then (D-CR9) is a way to evaluate if these properties are preserved by F.

B.1.2 Axioms on pre-distances between sets

Definition 37 ($\star$)**.** Let X be a set of sets and F a pre-distance on X. We define the following axioms:

($\mathbf{CR7}_{\mathtt{Far}}$) : (*Far decomposability*)
$\forall A, B \in X, \forall x \in \bigcup X,\ F(\{x\}, B) > F(A, B) \Rightarrow F(A \cup \{x\}, B) = F(A, B) + F(\{x\}, B)$

(CR9) : (Δ-*monotony*)
$\forall A, A', B, B' \in X,\ \#(A \Delta B) < \#(A' \Delta B') \Rightarrow F(A, B) < F(A', B')$

(CR18) : (*Intersection indifference*)
$\forall A, B \in X,\ F(A, B) = F(A \setminus B, B \setminus A)$

B.1.3 Axioms on pre-distances between subsets of a finite metric space

Definition 38 ($\star$)**.** Let X be a set. We define the following predicate `Far`:
Let $x \in X$, $A \subseteq X$, $B \subseteq X$, d a pre-distance on X:

$$\mathtt{Far}(x, A, B, d) \iff \forall y \in A, \forall z \in B, d(y, z) < d(x, z)$$

$\mathtt{Far}(x, A, B, d)$ means that x is further away from B than A is far from B, element by element.

Definition 39 ($\star$)**.** Let $\langle X, d\rangle$ be a finite metric space. We define the following axioms:

(d-$\mathbf{CR4}_W$) : (*Weak elementary d-monotony*)

$$\begin{gathered}\forall (x, x', y, y') \in X^4,\ \forall A \subseteq X, \forall B \subseteq X,\\ (\#\{x, x', y, y'\} = 4 \wedge \{x, x', y, y'\} \cap (A \cup B) = \emptyset \wedge d(x, x') < d(y, y'))\\ \Rightarrow F(A \cup \{x\}, B \cup \{x'\}) < F(A \cup \{y\}, B \cup \{y'\})\end{gathered}$$

(d-$\mathbf{CR4}_{\mathtt{Far}}$) : (*Far (elementary) d-monotony*)

$$\forall (x, x', y, y') \in X^4,\ \forall A \subseteq X, \forall B \subseteq X,$$
$$\left.\begin{array}{l} d(x, x') < d(y, y') \\ \mathtt{Far}(y, A \cup \{x\}, B \cup \{x', y'\}) \\ \mathtt{Far}(y', B \cup \{x'\}, A \cup \{x, y\}) \end{array}\right\} \Rightarrow F(A \cup \{x\}, B \cup \{x'\}) < F(A \cup \{y\}, B \cup \{y'\})$$

(d-CR10) : (*Global d-monotony*)
$\forall (A, A', B, B') \in (\mathtt{Pow}(X))^4,\ \displaystyle\sum_{(a,b)\in A\times B} d(a, b) < \sum_{(a,b)\in A'\times B'} d(a, b) \Rightarrow F(A, B) < F(A', B')$

(d-CR15) : (*d-match-represented*)
$\forall A \subseteq X, \forall B \subseteq X,$
$\exists A' \subseteq A, \exists B' \subseteq B$ such that $\#A' = \#B' = \min(\#A, \#B)$ and we can number the elements of A' and B' so that $F(A, B) = \displaystyle\sum_{i=1}^{\#A'} d(a_i, b_i)$

(d-CR17b) : (d*-void maximality*)
$\forall a \in X,\ F(\{a\}, \emptyset) = \mathtt{Dmax}(X, d)$

B.1.4 Axioms on pre-distances on cartesian products

This subsection allows us to study pre-distances between pairs and tuples of a cartesian product between sets, considering that we already know a pre-distance on each set, or on one of them.

Definition 40 ($\star$)**.** Let $\langle X, d\rangle$ be a finite metric space and Y be a set. We define:

(d-CR12S$_{\mathtt{L}}$) / (CR12S$_{\mathtt{R}}$) / (CR12S$_n$) : (*Strong left-monotony, strong right-monotony, strong n-monotony*).
(*Strong left-monotony*):
$\forall (x, x', y, y') \in X^4,\ \forall (a, a', b, b') \in Y^4,\ d(x, x') < d(y, y') \Rightarrow F(\langle x, a\rangle, \langle x', a'\rangle) < F(\langle y, b\rangle, \langle y', b'\rangle)$

B.1.5 Axioms on pre-distances between pairs of type (element,set)

The axioms of this section will specifically be useful to study pre-distances between S5 Kripke semantics, as we will see it in Section 4.

Definition 41 ($\star$)**.** Let X be a set of sets. We define:

Let D be a pre-distance on X,
(D-CR13) : (*Union* D*-monotony*)
$\forall (A, A', B, B') \in X^4,\ \forall (x, x', y, y') \in (\cup X)^4,$
$D(A \cup \{x\}, B \cup \{y\}) < D(A' \cup \{x'\}, B' \cup \{y'\}) \Rightarrow F(\langle x, A\rangle, \langle y, B\rangle) < F(\langle x', A'\rangle, \langle y', B'\rangle)$

(CR14) : (*Membership distinction*)
$\forall (A, B) \in X^2,\ \forall (x, x', y, y') \in (\bigcup X)^4,$

$$x \in A, x' \notin A, y \in B, y' \notin B \Rightarrow \begin{cases} F(\langle x, A\rangle, \langle y, B\rangle) < F(\langle x', A\rangle, \langle y, B\rangle) \\ F(\langle x, A\rangle, \langle y, B\rangle) < F(\langle x, A\rangle, \langle y', B\rangle) \\ F(\langle x', A\rangle, \langle y', B\rangle) < F(\langle x', A\rangle, \langle y, B\rangle) \\ F(\langle x', A\rangle, \langle y', B\rangle) < F(\langle x, A\rangle, \langle y', B\rangle) \end{cases}$$

B.1.6 Relations between axioms

Proposition 17 ($\star$)**.** *We have the following relations between axioms:*

Implications:

- (CR9) $\Rightarrow$ (CR16)
- (d-CR12S$_X$) $\Rightarrow$ (d-CR12$_X$) *(with* X *= L,R or n)*

Independences:

- (d-CR12$_X$) $\not\Rightarrow$ (d-CR12S$_X$) *(with* X *= L,R or n)*
- (CR16) $\not\Rightarrow$ (CR9)

Incompatibilities:

- *Let F be a function, F can satisfy at the most one strong monotony:* (d$_1$-CR12S$_{\mathtt{L}}$) ,(d$_2$-CR12S$_{\mathtt{R}}$) *or* (d$_n$-CR12S$_n$) *with an unique n.*
- (CR7$_{\mathtt{Str}}$) $\Rightarrow \neg$(d-CR17b)
- (CR1) + (CR2) + (CR8) $\Rightarrow \neg$(CR7$_{\mathtt{Far}}$)

Compatibilities:

- (CR12$_\mathtt{L}$) *and* (CR12$_\mathtt{R}$) *are compatible. As well as* (CR12$_n$) *and* (CR12$_m$) *with* $n \neq m$.
- (CR1)*,* (CR2)*,* (CR3)*,* (CR5)*,* (CR6)*,* (CR8)*,* (CR9)*,* (CR16) *and* (CR18) *are compatible.*
- (CR1)*,* (CR2)*,* (CR3)*,* (CR11)*,* (CR17a) *and* (d-CR17b) *are compatible.*
- (CR1)*,* (CR2)*,* (CR3)*,* (CR5)*,* (CR17a) *and* (d-CR17b) *are compatible.*

(The proof is to be found with the proofs of Section 2)

B.2 Further study of the impossibility result on (CR4)

Proposition 18 ($\star$)**.** ($\mathtt{Eucl}$-CR4$_W$) *cannot be satisfied (if* $\mathtt{Eucl}$ *is the euclidian distance in a geometric space).* ($\mathtt{Ham}$-CR4$_W$) *cannot be satisfied.*

Proof. We show that if F satisfy (d-CR4$_\mathtt{W}$) for some distance d on X, then we can find a counter example similar as the one for (d-CR4).

Let $\langle X, d$ be a metric space, suppose F satisfies (d-CR4$_\mathtt{W}$), let $x, y, y' \in X$ all distinct such that $d(x, x') < d(x, y')$ and such that there exists z and z' such that:

$$|\{x, x', y', z, z'\}| = 5 \text{ and } d(x, x') < d(z, z') \text{ and } d(z, z') < d(x, y').$$

Let $A, B \subseteq X$ such that $\{x, x', y', z, z'\} \cap (A \cup B) = \emptyset$.

Then $d(x, x') < d(z, z')$ gives $F(A \cup \{x\}, B \cup \{x'\}) < F(A \cup \{z\}, B \cup \{z'\})$

And $d(z, z') < d(x, y')$ gives $F(A \cup \{z\}, B \cup \{z'\}) < F(A \cup \{x\}, B \cup \{y'\})$

So, by transitivity of "$<$", we have $F(A \cup \{x\}, B \cup \{x'\}) < F(A \cup \{x\}, B \cup \{y'\})$.

A soon as such z and z' exist, we then have:

$$d(x, x') < d(x, y') \Rightarrow F(A \cup \{x\}, B \cup \{x'\}) < F(A \cup \{x\}, B \cup \{y'\})$$

We find some valuations to have a counter-example for (Ham-CR4$_\mathtt{W}$):

We define $x = (0, 0, 0)$, $x' = (0, 0, 1)$, $y' = (1, 1, 1)$. We then have: $\mathtt{Ham}(x, x') = 1 < 3 = \mathtt{Ham}(x, y')$. So we want: z and z' such that: $1 < \mathtt{Ham}(z, z') < 3$, ie. $\mathtt{Ham}(z, z') = 2$. $z = (1, 0, 0)$, $z' = (0, 1, 0)$ suits.

Finally, with $A = \{1, 1, 0\}$ $B = \{0, 1, 1\}$, we then have:

$(|\{x, x', z, z'\}| = 4 \wedge \{x, x', z, z'\} \cap (A \cup B) = \emptyset \wedge d(x, x') < d(z, z'))$ so $F(A \cup \{x\}, B \cup \{x'\}) < F(A \cup \{z\}, B \cup \{z'\})$.

And: $(|\{x, y', z, z'\}| = 4 \wedge \{x, y', z, z'\} \cap (A \cup B) = \emptyset \wedge d(z, z') < d(x, y'))$ so $F(A \cup \{z\}, B \cup \{z'\}) < F(A \cup \{x\}, B \cup \{y'\})$.

We reach a contradiction again.

$\square$

Remark 4 : We will always consider $\#X \geqslant 3$. If $\#X < 3$, (d-CR4) can be satified easily. In cases $\#X = 0$ and $\#X = 1$, $d(x, y) > 0$ never occurs, so neither does $d(x, x') < d(y, y')$. (d-CR4) is then trivially verified for any distance d on X. In the case $\#X = 2$, $d(x, x') < d(y, y')$ can not occur either.

Remark 5 : In fact the satisfiability of (d-CR4) strongly depends on X. We first consider that we can take any point in X that we need to find a counter-example. We later specify the conditions on X to reach the impossibility of (d-CR4).

I tried to generalise this result to more distances in 2 different manners:

Firstly, by generalising the previous proof:

Definition 42 ($\star$)**.** Let $\langle X, d\rangle$ be a metric space, let x_1, x_2, x_3, x_4 be elements of X. If $d(x_1, x_3) < d(x_1, x_4)$ and $d(x_2, x_3) > d(x_2, x_4)$, we call $\{x_1, x_2, x_3, x_4\}$ a *lozenge* of X.

Proposition 19 ($\star$)**.** *Let $\langle X, d\rangle$ be a metric space. If $\langle X, d\rangle$ has a lozenge, then* (d-CR4) *cannot be satisfied.*

Proof. This is exactly the Remark 3. ☐

Secondly I used the intuition that CR4 cannot be satisfied in a space of dimension two or more, because our previous counter-example will still apply. In order to do this, we need to define a dimension for finite metric spaces.

Definition 43 ($\star$)**.** We name *grid* of dimension n and length m the set $\mathtt{G}^n_m$ of all the vectors of dimension n with all elements in $[\![1, m]\!]$. An element of $\mathtt{G}^n_m$ is then called a *point* of the grid.

Definition 44 ($\star$)**.** Let n, m be two natural numbers, let $\mathbf{v}$ be a vector of n elements of $(0, \infty)$. We define the *extended manhattan distance* on $\mathtt{G}^n_m$ as

$$\mathtt{ExMan}_{\mathbf{v}} : \left| \begin{array}{ccc} (\mathtt{G}^n_m)^2 & \longrightarrow & [0, \infty) \\ (g, h) & \longmapsto & \displaystyle\sum_{i=1}^{n} \mathbf{v}_i \times |g_i - h_i| \end{array} \right.$$

I then inject the metric space into the grid structure, so that I can easely transfer the nice properties of the grid to the metric space.

Definition 45 ($\star$)**.** Let $\langle X, d\rangle$ be a metric space. We say that $\langle X, d\rangle$ is *straight* if there exist $n, m \in \mathbb{N}$ such that there exists a surjective function f from X to $\mathtt{G}^n_m$ and a strictly positive vector $\mathbf{v}$ such that $\forall (x, y) \in X^2, d(x, y) = \mathtt{ExMan}_{\mathbf{v}}(f(x), f(y))$.
Besides, if for all $n' > n$, there is no surjective function from X to $\mathtt{G}^{n'}_{m'}$ for any m', then n is the *dimension* of $\langle X, d\rangle$.
If $\mathbf{v} = k \times \mathbf{1}$ with $k \in \mathbb{R}$, $\langle X, d\rangle$ is called *regular.*

Proposition 20 ($\star$)**.** *Let $\langle X, d\rangle$ be a metric space. If, $\langle X, d\rangle$ is straight, then* (d-CR4) *can be satisfied if and only if $\langle X, d\rangle$ is of dimension 2 or more.*

Proof. If $\langle X, d\rangle$ is a straight metric space of dimension $n \geq 2$, then for all $m \geq 1$, $\mathtt{G}^n_m$ has a lozenge, and so does $\langle X, d\rangle$. ☐

B.3 Other pre-distances

I studied other pre-distances in order to find pre-distances with specific properties.

Definition 46 ($\star$)**.** Let $\langle X, d\rangle$ be a metric space. We define the following functions on $\mathtt{Pow}(X) \times \mathtt{Pow}(X)$, such that $\forall A, B \in \mathtt{Pow}(X)$: (with $\min(\emptyset) = 0$)

- $\mathtt{Nij}_d(A, B) \triangleq |\#A - \#B| + \begin{cases} \min\limits_{f:A\hookrightarrow B} \mathtt{Dist}(d, A, B, f) \text{ si } |A| \leqslant |B| \\ \min\limits_{f:B\hookrightarrow A} \mathtt{Dist}(d, B, A, f) \text{ otherwise} \end{cases}$

Definition 47 ($\star$)**.** Let $\langle X, d\rangle$ be a metric space. We define the following functions on $\mathtt{Pow}(X) \times \mathtt{Pow}(X)$, such that $\forall A, B \in \mathtt{Pow}(X)$: (with $\min(\emptyset) = \mathtt{Dmax}(X, d)$)

- $\mathtt{WSD}_d(A, B) \triangleq \displaystyle\sum_{(x,y)\in(A\Delta B)^2} d(x, y) + \mathtt{Drast}(A, B)$ (*Weighted Symmetric Difference*)

- $\mathtt{MaxSum}_d(A, B) \triangleq \displaystyle\sum_{(x,y)\in A\times B} d(x, y)$ (*Maximal Sum*)

- $\mathtt{Max}_d(A,B) \triangleq \max_{(x,y)\in A\times B} d(x,y)$

Proposition 21 ($\star$)**.** *Let $\langle X, d\rangle$ be a metric space, $\mathtt{Nij}_d, \mathtt{WSD}_d, \mathtt{MaxSum}_d, \mathtt{Max}_d$ are not distances.*

Proof. ($\mathtt{Nij}_d$ *is identity respecting, symmetric but does not respect the triangle inequality*)

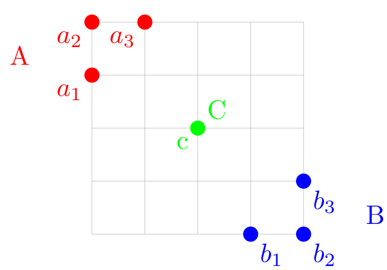
Let $\langle X, d\rangle$ be a metric space and $A, B, C \subseteq X$.

- Suppose $A = B$, then $|A| \leqslant |B|$ so
$$\begin{aligned} \mathtt{Nij}_d(A,B) &= \min_{f:A\hookrightarrow B} \mathtt{Dist}(d,A,B,f) \\ &= \mathtt{Dist}(\mathtt{id},A,B,f) \\ &= \sum_{a\in A} d(a,a) + |B| - |A| \\ &= 0 \end{aligned}$$
Reciprocally, suppose $\mathtt{Nij}_d(A,B) = 0$, then:
  - Either $|A| > |B|$ and $\exists f : B \hookrightarrow A \mid \mathtt{Dist}(d,B,A,f) = 0$, so $|A| - |B| = 0$ so $|B| = |A|$. Contradiction. This cannot append.
  - Or $|A| \leqslant |B|$ and $\exists f : A \hookrightarrow B \mid \mathtt{Dist}(d,A,B,f) = 0$,
    so $|B| - |A| = 0$ so $|B| = |A|$,
    and $\sum_{a\in A} d(a, f(a)) = 0$ ie. $\forall a \in A, d(a, f(a)) = 0$ ie. $\forall a \in A, a = f(a)$ ie. $f = \mathtt{Id}$.
    So $B \subseteq A$ and $|B| = |A|$. So $A = B$.

- If $|A| \neq |B|$, then:
  - Either $|A| < |B|$, then $\mathtt{Nij}_d(A,B) = \min_{f:A\hookrightarrow B} \mathtt{Dist}(d,A,B,f) = \mathtt{Nij}_d(B,A)$
  - Or $|A| > |B|$, then $\mathtt{Nij}_d(A,B) = \min_{f:B\hookrightarrow A} \mathtt{Dist}(d,B,A,f) = \mathtt{Nij}_d(B,A)$

  If $|A| = |B|$: then the set of injectives functions from A to B is exactly the set of bijections from A to B, and the set of their inverse functions is exactly the set of bijections from B to A. Since d is a distance, d is symmetric, so $\min_{f:A\hookrightarrow B} \mathtt{Dist}(d,A,B,f) = \min_{f:B\hookrightarrow A} \mathtt{Dist}(d,B,A,f)$. Plus $|A| - |B| = 0 = |B| - |A|$.
  So $\mathtt{Nij}_d(A,B) = \min_{f:A\hookrightarrow B} \mathtt{Dist}(d,A,B,f) = \min_{f:B\hookrightarrow A} \mathtt{Dist}(d,B,A,f) = \mathtt{Nij}_d(B,A)$.

- In order to show that $\mathtt{Nij}_d$ does not respect the triangle inequality, we use the following counter-example, with X being a set of points in space and d being the Manhattan distance $\mathtt{Man}$ (number of segments between the two points):



  $\mathtt{Nij}_d(A,B) = 6 + 8 + 6 = 20$
  $\mathtt{Nij}_d(A,C) = 3 + 2 = 5$
  $\mathtt{Nij}_d(C,B) = 3 + 2 = 5$

  So $\mathtt{Nij}_d(A,B) > \mathtt{Nij}_d(A,C) + \mathtt{Nij}_d(C,B)$

$\square$

Proposition 22 ($\star$). *Let $\langle X, d\rangle$ be a metric space, $\texttt{Nij}_d$ is a distance between subsets of X with the same cardinality.*

Proof. Let $\langle X, d\rangle$ be a metric space. Let $A, B, C \subseteq X$. Suppose *wlog* $\#A \leqslant \#B$ (because $\texttt{Nij}_d$ is symmetric). Then we can be in three different casess:

- $\#A \leqslant \#B \leqslant \#C$
- $\#A \leqslant \#C \leqslant \#B$
- $\#C \leqslant \#A \leqslant \#B$

In the case where $\#A \leqslant \#C \leqslant \#B$, we have:

$\{\psi \circ \varphi \mid \varphi : A \hookrightarrow C, \psi : C \hookrightarrow B\} \subseteq \{f : A \hookrightarrow B\}$ (1)

$$\begin{aligned}
\texttt{Nij}_d(A, C) + \texttt{Nij}_d(C, B) &= \min_{\varphi:A\hookrightarrow C} \texttt{Dist}(d, A, C, \varphi) + \min_{\psi:C\hookrightarrow B} \texttt{Dist}(d, C, B, \psi) \\
&= \min_{\substack{\varphi\,:\,A\,\hookrightarrow\,C \\ \psi\,:\,C\,\hookrightarrow\,B}} (\sum_{a\in A} d(a, \varphi(a)) + \#C - \#A + \sum_{c\in C} d(c, \psi(c)) + \#B - \#C) \\
&\geqslant \min_{\substack{\varphi\,:\,A\,\hookrightarrow\,C \\ \psi\,:\,C\,\hookrightarrow\,B}} (\sum_{a\in A} d(a, \varphi(a)) + \sum_{c\in\varphi(A)} d(c, \psi(c))) + \#B - \#A \\
&= \min_{\substack{\varphi\,:\,A\,\hookrightarrow\,C \\ \psi\,:\,C\,\hookrightarrow\,B}} \sum_{a\in A} (d(a, \varphi(a)) + d(\varphi(a), \psi(\varphi(a))) + \#B - \#A \\
&\geqslant \min_{\substack{\varphi\,:\,A\,\hookrightarrow\,C \\ \psi\,:\,C\,\hookrightarrow\,B}} \sum_{a\in A} (d(a, \psi(\varphi(a))) + \#B - \#A \\
&\geqslant \min_{f:A\hookrightarrow B} \sum_{a\in A} d(a, f(a) + \#B - \#A \qquad (\text{ with } (1)\ ) \\
&= \texttt{Nij}_d(A, B)
\end{aligned}$$

Finally, in the special case where all the considered subsets of X have the same cardinality, we are always in the first case, so $\texttt{Nij}_d$ satisfies the triangle inequality for subsets of same cardinality.

□

Proof. ($\texttt{WSD}$ *is identity respecting, symmetric, but does respect the triangle inequality.*)
Let $\langle X, d\rangle$ be a metric space and $A, B, C \subseteq X$.

- Suppose $\texttt{WSD}_d(A, B) = 0$. All the summed terms are positives, so $\texttt{Drast}(A, B) = 0$, ie. $A = B$.
  Reciprocally, suppose $A = B$. Then $\texttt{Drast}(A, B) = 0$. Plus, $A\Delta B = \emptyset$ so the sum is empty. So $\texttt{WSD}_d(A, B) = 0$.
- $A\Delta B = B\Delta A$ and $\texttt{Drast}$ is symmetric because it is a distance.
  $$\begin{aligned}
  \texttt{WSD}_d(A, B) &= \sum_{(a,b)\in(A\Delta B)^2} d(a, b) + \texttt{Drast}(A, B) \\
  &= \sum_{(a,b)\in(B\Delta A)^2} d(a, b) + \texttt{Drast}(B, A) \\
  &= \texttt{WSD}_d(A, B)
  \end{aligned}$$
- Counter-example for the triangle inequality (with $d = \texttt{Ham}$ and $X = \texttt{Val}(\{p_1, p_2, p_3\})$):

  We define: $v = (0, 0, 0)$, $u = (0, 1, 0)$, $w = (0, 1, 1)$ and $U = \{u\}$, $V = \{u, v\}$, $W = \{u, w\}$.

  Then $\texttt{WSD}_{\texttt{Ham}}(V, W) = \texttt{Ham}(v, v) + \texttt{Ham}(w, w) + \texttt{Ham}(v, w) + \texttt{Ham}(w, v) + \texttt{Drast}(A, B) = 0 + 2 \times 2 + 1 = 5$
  And: $\texttt{WSD}_{\texttt{Ham}}(V, U) + \texttt{WSD}_{\texttt{Ham}}(U, W) = \texttt{Ham}(v, v) + \texttt{Drast}(V, U) + \texttt{Ham}(w, w) + drast(U, W) = 0 + 1 + 0 + 1 = 2$.

  So we have a case where $\texttt{WSD}_{\texttt{Ham}}(V, W) > \texttt{WSD}_{\texttt{Ham}}(V, U) + \texttt{WSD}_{\texttt{Ham}}(U, W)$.

□

Proof. ($\texttt{MaxSum}$ *is symmetric but it is neither identity respecting, nor does it respect the triangle inequality.*)

Let $\langle X, d\rangle$ be a metric space and $A, B \subseteq X$.

- $\texttt{MaxSum}_d(A,B) = \sum\limits_{(x,y)\in A\times B} d(x,y) = \sum\limits_{(x,y)\in A\times B} d(y,x) = \sum\limits_{(x,y)\in B\times A} d(x,y) = \texttt{MaxSum}_d(B,A)$
- Let X be $\{1,2,3\}$ and A be $\{1,2\}$, we define $\mathtt{d}(x,y) \triangleq |x-y|$, then $\texttt{MaxSum}_\mathtt{d}(A,A) = |1-1|+|1-2|+|2-1|+|2-2| = 2 > 0$.
- With the same example as below, $\texttt{MaxSum}_\mathtt{d}(A,A) = 2 > 0 = \texttt{MaxSum}_\mathtt{d}(A,\emptyset) + \texttt{MaxSum}_\mathtt{d}(\emptyset,A)$.

$\square$

Proof. ($\texttt{Max}$ *is symmetric and satisfies the triangle inequality but it is not identity respecting.*)

Let $\langle X,d\rangle$ be a metric space and $A,B,C \subseteq X$.

- $\texttt{Max}_d(A,B) = \max_{(a,b)\in A\times B} d(a,b) = \max_{(a,b)\in A\times B} d(b,a) = \max_{(a,b)\in B\times A} d(a,b) = \texttt{Max}_d(B,A)$.
- If A,B,C are not empty:
  $\texttt{Max}_d(A,C) + \texttt{Max}_d(C,B) = \max\limits_{(a,c)\in A\times C} d(a,c) + \max\limits_{(c,b)\in C\times B} d(c,b)$ and $\texttt{Max}_d(A,B) = \max\limits_{(a,b)\in A\times B} d(a,b)$ so there exist $a_1 \in A, b_1 \in B$ such that $\texttt{Max}_d(A,B) = d(a_1,b_1)$ and for all $c_1 \in C$, $\texttt{Max}_d(A,C) = \max\limits_{(a,c)\in A\times C} d(a,c) \geqslant d(a_1,c_1)$ and $\texttt{Max}_d(C,B) = \max\limits_{(c,b)\in C\times B} d(c,b) \geqslant d(c_1,b_1)$. So $\texttt{Max}_d(A,C) + \texttt{Max}_d(C,B) \geqslant d(a_1,c_1) + d(c_1,b_1) \geqslant d(a_1,b_1) = \texttt{Max}_d(A,B)$.

  If C is emply:
  $\texttt{Max}_d(A,C) + \texttt{Max}_d(C,B) = 2\texttt{Dmax}(X,d)$ and $\texttt{Max}_d(A,B) \geqslant \texttt{Dmax}(X,d)$ so $\texttt{Max}_d(A,C) + \texttt{Max}_d(C,B) \geqslant \texttt{Max}_d(A,B)$.

  If C is not empty but A is (or, the same way, if C is not empty but B is):
  $\texttt{Max}_d(A,C) + \texttt{Max}_d(C,B) = \texttt{Dmax}(X,d) + \texttt{Max}_d(C,B) \geqslant \texttt{Dmax}(X,d) \geqslant \texttt{Max}_d(A,B)$.
- Let X be $\{1,2,3\}$ and A be $\{1,2\}$, we define $\mathtt{d}(x,y) \triangleq |x-y|$, then $\texttt{Max}_\mathtt{d}(A,A) = |2-1| = 1 > 0$.

$\square$

B.4 Full axiomatic study and proofs of the axiomatic study

		Pre-distances		Pre-distances Between sets		Pre-distances between subsets of a metric space								Pre-distances Between Tuples
		Drast	Plus(d1,d2)	Delta	Bin	Inj(d)	Haus(d)	Nij(d)	PoinSet(d)	WSD(d)	MaxSum(d)	Max(d)	Pairs(d)	Sum(d,...)
Axioms on Pre-distances	CR1	V	V	V	V	V	V	V	V	V	X	X	X****	V
	CR2	V	V	V	V	V	V	V	V	V	V	V	V	V
	CR3	V	V	V	V	V	V	X***	X	X	X	V	X	V
	D-CR9	D	d, d1, d2	D		D								D
Axioms on Pre-distances Between sets	CR5	V	d1,d2	V	X	V	X	V	X	X	X	X	X	
	CR6	V		V	X	X	X	X	X	X	X	X	X	
	CR7 far	X		X	X	X	X	X	X	X		X		
	CR7 str	X		X	X	X	X	X	X	X	V	X	V	
	CR8 K	X		V	X	X	X	X	X	X	X	X	X	
	CR9	X		V	X	X	X	X	X	X	X	X	X	
	CR16	X		V	X	X	X	X	X	V	X	X	X	
	CR18	V		V	V		X			V	X	X	X	
Axioms on Pre-distances Between substes Of a metric space	d-CR4 far	d	d,d1,d2			V								
	d-CR4 far'											V		
	d-CR10			X	d	X	X	X	X	X	V	X		
	d-CR10'			X							V			
	d-CR11					X*	V	X	X	X	X	V	X	
	d-CR17a	S*(7)				V	V	V	V*(5)	X*(6)	V	V	V	
	d-CR17b	S*(8)				V	V	X	V	X	X	V	X*(9)	
Pre-distance Between tuples	d-CR12		d, d1, d2											V
	d-CR12S													d
Between elements of Type (element,set)	D-CR13	X	d1,d2											
	CR14	X												

Legend:	* : iff |X| > 2
several subcases : what it depends on	** : except when |X| < 3, then Delta satisfies Drast-CR11 and does not satify d-CR11 for any other distance d
not relevant	*** : satifies the triangle inequality for subsets of same cardinality
no result	****: satisfies CR1' : F(A,A) = 0
V = satisfies	(5) : not exactly : F1({x},{y}) = 2 d(x,y)
X = does not satisfy	(6) : almost : F2({x},{y}) = 2d(x,y) + drast(x,y)
S = special	(7) : satisfies iff d=drast
distance	(8) : satisfies iff Dmax = 1 ie d = drast
	(9) : Pairs(A,vide) = 0 for all A

Figure 7: Full axiomatic study of all the introduced pre-distances.

Proofs:

(CR1), (CR2) and (CR3) were already studied previously for each pre-distance.

B.4.1 (CR5)

Proof. ($\texttt{Drast}$ *satisfies* (CR5))
Let X be a set of sets, $A, A', B \in X$ such that $A \subseteq B$ and $A' \subseteq B$ and $\#A = \#A'$:
If $\#A = \#B$, then $A = A' = B$ so $\texttt{Drast}(A, B) = 0 = \texttt{Drast}(A', B)$.
If $\#A < \#B$ then $A \neq B$ and $\#A' < \#B$ so $A' \neq B$. So $\texttt{Drast}(A, B) = drast(A', B)$

$\square$

Proof. ($\texttt{Delta}$ *satisfies* (CR5))
Let X be a set of sets, $A, A', B \in X$ such that $A \subseteq B$ and $A' \subseteq B$ and $\#A = \#A'$:
$\texttt{Delta}(A, B) = \#(A \Delta B) = \#B - \#A = \#B - \#A' = \texttt{Delta}(A', B)$.

$\square$

Proof. ($\texttt{Bin}$ *does not satisfy* (CR5))
Let $X = \{x_1, x_2, x_3\}, B = X, A = \{x_1, x_2\}, A' = \{x_1, x_3\}, \#A = \#A'$:
$\texttt{Bin}(A, B) = 1 \times 4 + 1 \times 2 + 0 = 6$
$\texttt{Bin}(A', B) = 1 \times 4 + 0 + 1 \times 1 = 5 \neq \texttt{Bin}(A, B)$

$\square$

Proof. ($\texttt{Inj}$ *satisfies* (CR5))
Let X be a set of sets, d be a distance on $\bigcup X$, $A, A', B \in X$ such that $A \subseteq B$ and $A' \subseteq B$ and $\#A = \#A'$:
If $\#A < \#B$:

$$\begin{aligned}
\texttt{Inj}_d(A, B) &= \min_{f: A \hookrightarrow B} \texttt{Dist}(d, A, B, f) + \min_{f: B \hookrightarrow A} \texttt{Dist}(d, B, A, f) + |\#A - \#B| \times \texttt{Dmax}(X, d) \\
&= \texttt{Dist}(d, A, B, \texttt{Id}) + 0 + |\#A - \#B| \times \texttt{Dmax}(X, d) \\
&= |\#A - \#B| \times \texttt{Dmax}(X, d) \\
&= |\#A' - \#B| \times \texttt{Dmax}(X, d) \\
&= \texttt{Inj}_d(A', B)
\end{aligned}$$

If $\#A = \#B$ then $A = A' = B$ so $\texttt{Inj}_d(A, B) = 0 = \texttt{Inj}_d(A', B)$ because $\texttt{Inj}_d$ is a distance.

$\square$

Proof. ($\texttt{Haus}$ *does not satisfy* (CR5))
We use as counter-example: $A = \{1, 2\}, A' = \{1, 5\}, B = \{1, 2, 5\}, X = \{A, A', B\}, \mathtt{d}(n, m) \triangleq |n - m|$. $\texttt{Haus}_\mathtt{d}(A, B) = \max(\max_{a \in A} \min_{b \in B} \mathtt{d}(a, b), \max_{b \in B} \min_{a \in A} \mathtt{d}(a, b)) = \max\{0, 0, 0, 0, 3\} = 3$

$\texttt{Haus}_\mathtt{d}(A', B) = \max(\max_{a \in A'} \min_{b \in B} \mathtt{d}(a, b), \max_{b \in B} \min_{a \in A'} \mathtt{d}(a, b)) = \max\{0, 0, 0, 1, 0\} = 1 \neq \texttt{Haus}_\mathtt{d}(A, B)$ $\square$

Proof. ($\texttt{Nij}$ *satisfies* (CR5))

Let X be a set of sets, d be a distance on $\bigcup X$, $A, A', B \in X$ such that $A \subseteq B$ and $A' \subseteq B$ and $\#A = \#A'$:
If $\#A < \#B$:

$$\begin{aligned}
\texttt{Nij}_d(A, B) &= \min_{f: A \hookrightarrow B} \texttt{Dist}(d, A, B, f) + |\#A - \#B| \\
&= \texttt{Dist}(d, A, B, \texttt{Id}) + |\#A - \#B| \\
&= |\#A - \#B| \\
&= |\#A' - \#B| \\
&= \texttt{Nij}_d(A', B)
\end{aligned}$$

If $\#A = \#B$ then $A = A' = B$ so $\texttt{Nij}_d(A, B) = 0 = \texttt{Nij}_d(A', B)$ because $\texttt{Nij}_d$ satisfies (CR1).

$\square$

Proof. ($\texttt{PointSet}$ *does not satisfy* (CR5))
Same counter-example as for $\texttt{Haus}$. $\square$

Proof. ($\texttt{WSD}$ *does not satisfy* (CR5))
We use as counter-example: $A = \{2, 3\}, A' = \{1, 4\}, B = \{1, 2, 3, 4\}, X = \{A, A', B\}, \mathtt{d}(n, m) \triangleq |n - m|$.
$\texttt{PointSet}_\mathtt{d}(A, B) = |1 - 4| + |4 - 1| = 6 \neq 2 = |2 - 3| + |3 - 2| = \texttt{PointSet}_\mathtt{d}(A', B)$ $\square$

Proof. ($\texttt{MaxSum}$ *does not satisfy* (CR5))
Same counter-example as for $\texttt{Haus}$. $\square$

Proof. (Max *does not satisfy* (CR5))
We use as counter-example: $A = \{1\}, A' = \{2\}, B = \{1,2,3\}, X = \{A, A', B\}, \mathtt{d}(n,m) \triangleq |n - m|$: $\mathtt{Max_d}(A,B) = 2 \neq 1 = \mathtt{Max_d}(A', B)$. ☐

Proof. (Pairs *does not satisfy* (CR5))
Same counter-example as for Haus. ☐

B.4.2 (CR6)

Proof. (Drast *satisfies* (CR6))

Let X be a set of sets, $A, B \in X$, $x, y \in \bigcup X$ such that $\{x,y\} \cap (A \cup B) = \emptyset$:
$\mathtt{Drast}(A \cup \{x\}, B) = 1 = \mathtt{Drast}(A \cup \{y\}, B)$

☐

Proof. (Delta *satisfies* (CR6))

Let X be a set of sets, $A, B \in X$, $x, y \in \bigcup X$ such that $\{x,y\} \cap (A \cup B) = \emptyset$:

$$\begin{aligned} \mathtt{Delta}(A \cup \{x\}, B) &= \#((A \cup \{x\})\Delta B) \\ &= \#(A \Delta B) + 1 \quad \text{(because } x \notin A\Delta B \text{ and } x \in (A \cup \{x\})\Delta B) \\ &= \#((A \cup \{y\})\Delta B) \\ &= \mathtt{Delta}(A \cup \{y\}, B) \end{aligned}$$

☐

Counter-example 0: $A = \{1\}, B = \{2\}, x = 3, y = 4, X = \{A, B, \{1,3\}, \{1,4\}\}, \mathtt{d}(n,m) \triangleq |n - m|$.

Proof. (Bin *does not satisfy* (CR6))
Counter-example 0. ☐

Proof. (Haus *does not satisfy* (CR6))
Counter-example 0. ☐

Proof. (Nij *does not satisfy* (CR6))
Counter-example with: $A = \{10\}, B = \{2\}, x = 3, y = 4, X = \{A, B, \{1,3\}, \{1,4\}\}, \mathtt{d}(n,m) \triangleq |n - m|$. ☐

Proof. (PointSet *does not satisfy* (CR6))
Counter-example 0. ☐

Proof. (WSD *does not satisfy* (CR6))
Counter-example 0. ☐

Proof. (MaxSum *does not satisfy* (CR6))
Counter-example 0. ☐

Proof. (Max *does not satisfy* (CR6))
Counter-example 0. ☐

Proof. (Pairs *does not satisfy* (CR6))
Counter-example: $A = \{2\}, B = \{1\}, x = 3, y = 4, \mathtt{d}(n,m) \triangleq |n - m|, X = \{A, B, \{2,3\}, \{2,4\}$. ☐

B.4.3 ($\mathrm{CR7}_{\mathtt{Far}}$)

Counter-example 1:
$A = \{2\}, B = \{3\}, C = \{1,2\}, D\{1,4\}, X = \{A, B, C, D\}, x = 1, \mathtt{d}(n,m) \triangleq |n - m|, \mathtt{Dmax}(X, \mathtt{d}) = 4$.

Proof. (Drast *does not satisfy* ($\mathrm{CR7}_{\mathtt{Far}}$))
See counter-example 1. ☐

Proof. (Delta *does not satisfy* ($\mathrm{CR7}_{\mathtt{Far}}$))

Counter-example: $A = \emptyset, B = \{1,2\}, x = 3, X = \{A, B, \{x\}\}$:
$\mathtt{Delta}(A, B) = 2 < 3 = \mathtt{Delta}(\{x\}, B)$
$\mathtt{Delta}(A \cup \{x\}, B) = \mathtt{Delta}(\{x\}, B) = 3 \neq 2 + 3 = \mathtt{Delta}(A, B) + \mathtt{Delta}(\{x\}, B)$

☐

Proof. (Bin *does not satisfy* ($\text{CR7}_{\texttt{Far}}$))
Counter-example: $A = \{1,2\}, B = \{1,3\}, x = 4, X = \{A, B, \{1,2,4\}\}$ and $\bigcup X$ is ordonned with the usual order on $\mathbb{N}$. □

Proof. (Inj *does not satisfy* ($\text{CR7}_{\texttt{Far}}$))

We use as counter-example: $A = \{2\}, B = \{3\}, C = \{1,2\}, X = \{A, B, C\}, \mathtt{d}(x,y) \triangleq |x - y|$:
$\mathtt{Dmax}(X, \mathtt{d}) = 2$
$\mathtt{Inj}_{\mathtt{d}}(A, B) = 1 + 1 + 0 = 2$
$\mathtt{Inj}_{\mathtt{d}}(\{x\}, B) = 2 + 2 + 0 = 4$
So $\mathtt{Inj}_{\mathtt{d}}(A, B) < \mathtt{Inj}_{\mathtt{d}}(\{x\}, B)$ but: $\mathtt{Inj}_{\mathtt{d}}(A \cup \{x\}, B) = 1 + 0 + 2 \times 1 = 3 \neq \mathtt{Inj}_{\mathtt{d}}(A, B) + \mathtt{Inj}_{\mathtt{d}}(\{x\}, B)$

□

Proof. (Haus *does not satisfy* ($\text{CR7}_{\texttt{Far}}$))
See counter-example 1. □

Proof. (Nij *does not satisfy* ($\text{CR7}_{\texttt{Far}}$))
See counter-example 1. □

Proof. (PointSet *does not satisfy* ($\text{CR7}_{\texttt{Far}}$))
See counter-example 1. □

Proof. (WSD *does not satisfy* ($\text{CR7}_{\texttt{Far}}$))
See counter-example 1. □

Proof. (Max *does not satisfy* ($\text{CR7}_{\texttt{Far}}$))
See counter-example 1. □

B.4.4 ($\text{CR7}_{\texttt{Str}}$)

Proof. (Drast *does not satisfy* ($\text{CR7}_{\texttt{Str}}$))
See counter-example 1. □

Proof. (Delta *does not satisfy* ($\text{CR7}_{\texttt{Str}}$))
Counter-example: $A = \emptyset, B = \{1,2\}, x = 3, X = \{A, B, \{x\}\}$:
$\mathtt{Delta}(A, B) = 2$
$\mathtt{Delta}(\{x\}, B) = 3$
$\mathtt{Delta}(A \cup \{x\}, B) = \mathtt{Delta}(\{x\}, B) = 3 \neq 2 + 3 = \mathtt{Delta}(A, B) + \mathtt{Delta}(\{x\}, B)$

□

Proof. (Bin *does not satisfy* ($\text{CR7}_{\texttt{Str}}$))
Counter-example: $A = \{1,2\}, B = \{1,3\}, x = 4, X = \{A, B, \{1,2,4\}\}$ and $\bigcup X$ is ordonned with the usual order on $\mathbb{N}$. □

Proof. (Inj *does not satisfy* ($\text{CR7}_{\texttt{Str}}$))

We use as counter-example: $A = \{1\}, B = \{3\}, C = \{1,2\}, X = \{A, B, C\}, \mathtt{d}(x,y) \triangleq |x - y|$:
$\mathtt{Dmax}(X, \mathtt{d}) = 2$
$\mathtt{Inj}_{\mathtt{d}}(A, B) = 2 + 2 + 0 = 4$
$\mathtt{Inj}_{\mathtt{d}}(\{x\}, B) = 1 + 1 + 0 = 2$
$\mathtt{Inj}_{\mathtt{d}}(A \cup \{x\}, B) = 1 + 0 + 2 \times 1 = 3 \neq \mathtt{Inj}_{\mathtt{d}}(A, B) + \mathtt{Inj}_{\mathtt{d}}(\{x\}, B)$ □

Proof. (Haus *does not satisfy* ($\text{CR7}_{\texttt{Str}}$))
See counter-example 1. □

Proof. (Nij *does not satisfy* ($\text{CR7}_{\texttt{Str}}$))
See counter-example 1. □

Proof. (PointSet *does not satisfy* ($\text{CR7}_{\texttt{Str}}$))
See counter-example 1. □

Proof. (WSD *does not satisfy* ($\text{CR7}_{\texttt{Str}}$))
See counter-example 1. □

Proof. ($\texttt{MaxSum}$ *satisfies* ($\mathrm{CR7}_{\texttt{Str}}$))

Let X be a set of sets,d a distance on $\bigcup X$, $A, B \in X$ and $x \in \bigcup X$ such that $x \notin A \cup B$:

$$\begin{aligned}
\texttt{MaxSum}_d(A \cup \{x\}, B) &= \sum_{(a,b)\in(A\cup\{x\})\times B} d(a,b) \\
&= \sum_{(a,b)\in A\times B} d(a,b) + \sum_{(a,b)\in\{x\}\times B} d(a,b) \\
&= \texttt{MaxSum}_d(A, B) + \texttt{MaxSum}_d(\{x\}, B)
\end{aligned}$$

□

Proof. ($\texttt{Max}$ *does not satisfy* ($\mathrm{CR7}_{\texttt{Str}}$))
See counter-example 1. □

Proof. ($\texttt{Pairs}$ *satisfies* ($\mathrm{CR7}_{\texttt{Str}}$))
See proposition 14. □

B.4.5 ($\mathrm{CR8}_K$)

Proof. ($\texttt{Drast}$ *does not satisfy* ($\mathrm{CR8}_K$) *for any* K)
$\texttt{Drast}$ is a distance so it cannot satisfy ($\mathrm{CR8}_K$) for any K by proposition 13. □

Proof. ($\texttt{Delta}$ *satisfies* ($\mathrm{CR8}_K$) *for* $K = 1$ *and no other* K)
Let X be a set of sets, $A, B \in X$, $x \in \bigcup X$ such that $x \notin A \cup B$. Then: $(A \cup \{x\})\Delta B = (A\Delta B)\dot{\cup}\{x\}$. So $\texttt{Delta}(A \cup \{x\}, B) = \#((A \cup \{x\})\Delta B) = \#(A\Delta B) + \#\{x\} = \texttt{Delta}(A, B) + 1$.
So $\texttt{Delta}$ satifies ($\mathrm{CR8}_K$) for $K = 1$ and no other K.

□

Proof. ($\texttt{Bin}$ *does not satisfy* ($\mathrm{CR8}_K$) *for any* K)
$\texttt{Bin}$ is a distance so it cannot satisfy ($\mathrm{CR8}_K$) for any K by proposition 13. □

Proof. ($\texttt{Inj}$ *does not satisfy* ($\mathrm{CR8}_K$) *for any* K)
Let $\langle X, d\rangle$ be a metric space, $\texttt{Inj}_d$ is a distance so it cannot satisfy ($\mathrm{CR8}_K$) for any K by proposition 13. □

Proof. ($\texttt{Haus}$ *does not satisfy* ($\mathrm{CR8}_K$) *for any* K)
Let $\langle X, d\rangle$ be a metric space, $\texttt{Haus}_d$ is a distance so it cannot satisfy ($\mathrm{CR8}_K$) for any K by proposition 13. □

Proof. ($\texttt{Nij}$ *does not satisfy* ($\mathrm{CR8}_K$) *for any* K)
Let X be $\{1, 2, 3, 4\}$ and $\mathtt{d}(n, m) \triangleq |n - m|$. Suppose that $\texttt{Nij}_\mathtt{d}$ satisfies ($\mathrm{CR8}_K$) for some K in $\mathbb{R}$. Then:
$\texttt{Nij}_\mathtt{d}(\{1\}, \{3\}) = |1 - 3| + 0 = 2$
$\texttt{Nij}_\mathtt{d}(\{1, 2\}, \{3\}) = |3 - 2| + 1 = 2 = \texttt{Nij}_\mathtt{d}(\{1\}, \{3\}) + 0$
So $K = 0$. And:
$\texttt{Nij}_\mathtt{d}(\{1\}, \{4\}) = |1 - 4| + 0 = 3$
$\texttt{Nij}_\mathtt{d}(\{1, 2\}, \{4\}) = |4 - 2| + 1 = 3 = \texttt{Nij}_\mathtt{d}(\{1\}, \{3\}) + 1$
So $K = 1$. Contradiction.

□

Proof. ($\texttt{PointSet}$ *does not satisfy* ($\mathrm{CR8}_K$) *for any* K)
Same counter-example as for $\texttt{Nij}$. □

Proof. ($\texttt{WSD}$ *does not satisfy* ($\mathrm{CR8}_K$) *for any* K)
Let X be $\{1, 2, 3, 4\}$ and $\mathtt{d}(n, m) \triangleq |n - m|$. Suppose that $\texttt{WSD}_\mathtt{d}$ satisfies ($\mathrm{CR8}_K$) for some K in $\mathbb{R}$. Then:
$\texttt{WSD}_\mathtt{d}(\{1\}, \{2\}) = |1 - 2| + |2 - 1| = 2$
$\texttt{WSD}_\mathtt{d}(\{1, 3\}, \{2\}) = (|1 - 2| + |1 - 3| + |2 - 3|) \times 2 = 8 = \texttt{WSD}_\mathtt{d}(\{1\}, \{2\}) + 6$
So $K = 6$. And:
$\texttt{WSD}_\mathtt{d}(\{1, 4\}, \{2\}) = (|1 - 2| + |2 - 4| + |1 - 4|) \times 2 = 12 = \texttt{Nij}_\mathtt{d}(\{1\}, \{2\}) + 10$
So $K = 10$. Contradiction. □

Proof. ($\texttt{MaxSum}$ *does not satisfy* ($\mathrm{CR8}_K$) *for any* K)
Same counter-example as for $\texttt{Nij}$. □

Proof. ($\texttt{Max}$ *does not satisfy* ($\mathrm{CR8}_K$) *for any* K)
Same counter-example as for $\texttt{WSD}$.

□

Proof. (`Pairs` *does not satisfy* (CR8$_K$) *for any* K)
Let X be a set of sets and d be a distance on $\bigcup X$. Suppose that $\texttt{Pairs}_d$ satisfies (CR8$_K$) for some K, then $\texttt{Pairs}_d$ satisfies (CR6) because (CR8$_K$) implies (CR6). However $\texttt{Pairs}_d$ does not satisfy (CR6). Contradiction. □

B.4.6 (CR16)

Proof. (`Drast` *does not satisfy* (CR16))
We use as counter-example: $A = \{1\}, B = \{3\}, C = \{1, 2\}, X = \{A, B, C\}, \mathtt{d}(x, y) \triangleq |x - y|$:
$\texttt{Drast}(A, B) = 1 = \texttt{Drast}(C, B)$ □

Proof. (`Delta` *satisfies* (CR16))
`Delta` satifies (CR8$_1$) so it satisfies (CR16) by proposition 1. □

Proof. (`Bin` *does not satisfy* (CR16))
We use as counter-example: $A = \{3\}, B = \{1\}, C = \{2, 3\}, X = \{A, B, C\}, \mathtt{d}(x, y) \triangleq |x-y|$ and the elements of $\bigcup X$ are ordonned by the usual order on $\mathbb{N}$. Then A is represented by 001, B by 100 and C by 011. So: $\texttt{Bin}(A, B) = |1 - 4| = 3 > 1 = |3 - 4| = \texttt{Bin}(C, B)$. □

Proof. (`Inj` *does not satisfy* (CR16))
Same counter-example as for `Drast`. □

Proof. (`Haus` *does not satisfy* (CR16))
Same counter-example as for `Drast`. □

Proof. (`Nij` *does not satisfy* (CR16))
Same counter-example as for `Drast`. □

Proof. (`PointSet` *does not satisfy* (CR16))

We use as counter-example: $A = \{1\}, B = \{3\}, C = \{1, 2\}, X = \{A, B, C\}, \mathtt{d}(x, y) \triangleq |x - y|$:
$\texttt{PointSet}_{\mathtt{d}}(A, B) = |1 - 3| + |3 - 1| = 4$
$\texttt{PointSet}_{\mathtt{d}}(A \cup \{2\}, B) = |1 - 3| + |2 - 3| + |3 - 2| = 4 = \texttt{PointSet}_{\mathtt{d}}(A, B)$.

□

Proof. (`WSD` *satisfies* (CR16))
Let X be a set of sets and d be a distance on $\bigcup X$. Let $A, B \in X$ and $x \in \bigcup X$ such that $x \notin A \cup B$ and $A \cup \{x\} \in X$:
If $A = B$:
$$\texttt{WSD}_d(A \cup \{x\}, B) = \sum_{(a,b)\in((A\cup\{x\})\Delta B)^2} d(a, b) + \texttt{Drast}(A \cup \{x\}, B) = \sum_{(a,b)\in(\{x\}^2} d(a, b) + 1 = 1 > 0 = \texttt{WSD}_d(A, B)$$
If $A \neq B$:
$$\begin{array}{rcl} \texttt{WSD}_d(A \cup \{x\}, B) & = & \displaystyle\sum_{(a,b)\in((A\cup\{x\})\Delta B)^2} d(a, b) + \texttt{Drast}(A \cup \{x\}, B) \\ & = & \displaystyle\sum_{(a,b)\in((A\Delta B)\dot{\cup}\{x\})^2} d(a, b) + 1 \\ & = & \displaystyle\sum_{(a,b)\in A\Delta B} d(a, b) + 2 \times \sum (a, b) \in \{x\} \times (A\Delta B) d(a, b) + d(x, x) + 1 \\ & = & \texttt{WSD}_d(A, B) + 2 \times \sum (a, b) \in \{x\} \times (A\Delta B) d(a, b) \\ & > & \texttt{WSD}_d(A, B) \end{array}$$
(Because $A\Delta B \neq \emptyset$ and $x \notin A\Delta B$.)

□

Proof. (`MaxSum` *does not satisfy* (CR16))
We use as counter-example: $A = B = \emptyset, C = \{1\}, X = \{A, C\}, \mathtt{d}(x, y) \triangleq |x - y|$:
$\texttt{MaxSum}_{\mathtt{d}}(A, B) = 0 = \texttt{MaxSum}_{\mathtt{d}}(C, B)$ □

Proof. (`Max` *does not satisfy* (CR16))

Let $A = \{1\}, B = \{3\}, x = 2, X = \{A, B, \{1, 2\}\}, \mathtt{d}(n, m) \triangleq |n - m|$.
$\texttt{Max}_{\mathtt{d}}(A \cup \{x\}, B) = |3 - 1| = \texttt{Max}_{\mathtt{d}}(A, B)$

□

Proof. ($\texttt{Pairs}$ *does not satisfy* (CR16))

Let $A = \{1\}, B = \emptyset, x = 2, X = \{A, B, \{x\}\}, \mathtt{d}(n,m) \triangleq |n-m|$.
$x \notin A \cup B$ and: $\texttt{Pairs}_\mathtt{d}(A \cup \{x\}, B) = \texttt{Pairs}_\mathtt{d}(A \cup \{x\}, \emptyset) = 0 = \texttt{Pairs}_\mathtt{d}(A, B)$.

□

B.4.7 (CR18)

Proof. ($\texttt{Drast}$ *satisfies* (CR18))

Let X be a set of sets, $A, B \in X$: $A = (A \setminus B) \cup (A \cap B)$ and $B = (B \setminus A) \cup (A \cap B)$, so $A = B \iff A \setminus B = B \setminus A$. So: $\texttt{Drast}(A,B) = 1 \iff A = B \iff A \setminus B = B \setminus A \iff \texttt{Drast}(A \setminus B, B \setminus A) = 1$. The same: $\texttt{Drast}(A,B) = 0 \iff \texttt{Drast}(A \setminus B, B \setminus A) = 0$.
Since $\texttt{Drast} \in \{0,1\}^X$, $\texttt{Drast}(A,B) = \texttt{Drast}(A \setminus B, B \setminus A)$.

□

Proof. ($\texttt{Delta}$ *satisfies* (CR18))

Let X be a set of sets, $A, B \in X$: $\texttt{Delta}(A,B) = \#(A \Delta B) = \#((A \setminus B) \Delta (B \setminus A)) = \texttt{Delta}(A \setminus B, B \setminus A)$. □

Proof. ($\texttt{Bin}$ *satisfies* (CR18))

Let X be a set of sets, $A, B \in X$, suppose (*wlog*) that the elements of $\bigcup X$ are ordonned ($\bigcup X = \{x_1, x_2, ...\}$):

$$\begin{aligned}
\texttt{Bin}(A,B) &= |\texttt{Numb}(A) - \texttt{Numb}(B)| \\
&= |\sum_{x_k \in A} 2^k - \sum_{x_k \in B} 2^k| \\
&= |\sum_{x_k \in A \cap B} 2^k + \sum_{x_k \in A \setminus B} 2^k - \sum_{x_k \in A \cap B} 2^k - \sum_{x_k \in B \setminus A} 2^k| \\
&= |\sum_{x_k \in A \setminus B} 2^k - \sum_{x_k \in B \setminus A} 2^k| \\
&= \texttt{Bin}(A \setminus B, B \setminus A)
\end{aligned}$$

□

Proof. ($\texttt{Haus}$ *does not satisfy* (CR18))

Let X be a set of sets, d be a distance on $\bigcup X$, $A, B \in X$, suppose that $B \subseteq A$ and $\texttt{Dmax}(A, d) < \texttt{Dmax}(\bigcup X, d)$.:
(for example $A = \{1,2,3\}, B = \{1\}, C = \{7,8\}, X = \{A, B, C\}$ and $\mathtt{d}$ is the distance on $\mathbb{N}$ defined by: $\mathtt{d}(x,y) \triangleq |x-y|$)
Then $\texttt{Haus}_\mathtt{d}(A,B) \leqslant \texttt{Dmax}(A,d) < \texttt{Dmax}(\bigcup X, d)$ and:
$\texttt{Haus}_\mathtt{d}(A \setminus B, B \setminus A) = \texttt{Haus}_\mathtt{d}(A \setminus B, \emptyset) = \texttt{Dmax}(\bigcup X, d)$.
So we have a case where: $\texttt{Haus}_\mathtt{d}(A,B) \neq \texttt{Haus}_\mathtt{d}(A \setminus B, B \setminus A)$.

□

Proof. ($\texttt{WSD}$ *satisfies* (CR18))

Let X be a set of sets, d be a distance on $\bigcup X$, $A, B \in X$: $A \Delta B = (A \setminus B) \Delta (B \setminus A)$ and $\texttt{Drast}(A,B) = \texttt{Drast}(A \setminus B, B \setminus A)$ (see the first proof of this subsection). So:

$$\begin{aligned}
\texttt{WSD}_d(A,B) &= \sum_{(x,y) \in (A \Delta B)^2} d(x,y) + \texttt{Drast}(A,B) \\
&= \sum_{(x,y) \in ((A \setminus B) \Delta (B \setminus A))^2} d(x,y) + \texttt{Drast}(A \setminus B, B \setminus A) \\
&= \texttt{WSD}_d(A \setminus B, B \setminus A)
\end{aligned}$$

□

Proof. ($\texttt{MaxSum}$ *does not satisfy* (CR18))

We use the following counter-example: $A = \{1,2\}, B = \{2,3\}, X = \{A, B\}$ and $\mathtt{d}$ defined as $\mathtt{d}(x,y) = |x-y|$.
$\texttt{MaxSum}_\mathtt{d}(A,B) = |1-2| + |1-3| + |2-2| + |2-3| = 4$
$\texttt{MaxSum}_\mathtt{d}(A \setminus B, B \setminus A) = |1-3| = 2$
So we have a case where: $\texttt{MaxSum}_\mathtt{d}(A,B) \neq \texttt{MaxSum}_\mathtt{d}(A \setminus B, B \setminus A)$. □

Proof. (`Max` *does not satisfy* (CR18))
We use the following counter-example: $A = \{1, 2, 9\}, B = \{1, 3, 9\}, X = \{A, B\}$ and $\mathtt{d}$ defined as $\mathtt{d}(x, y) = |x - y|$.
Then: $\mathtt{Max}_{\mathtt{d}}(A, B) = |1 - 9| = 8 \neq 1 = |2 - 3| = \mathtt{Max}_{\mathtt{d}}(A \setminus B, B \setminus A)$

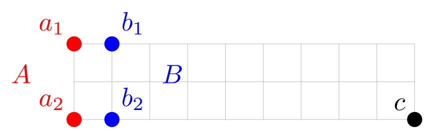
□

Proof. (`Pairs` *does not satisfy* (CR18))
Same counter-example as the proof for `MaxSum`:
$\mathtt{Pairs}_{\mathtt{d}}(A, B) = |1 - 2| + |1 - 3| + |2 - 2| + |2 - 3| - |2 - 2| = 4$
$\mathtt{Pairs}_{\mathtt{d}}(A \setminus B, B \setminus A) = |1 - 3| = 2 \neq \mathtt{Pairs}_{\mathtt{d}}(A, B)$.

□

B.4.8 (d-CR11)

Counter-example 2:



$X = \{A, B, \{c\}\}$. We use `Man` as a distance on $\bigcup X$. $\mathtt{Dmax}(x, \mathtt{Man}) = 10$.

Proof. ($\mathtt{Inj}_d$ *satisfies* (d-CR11) *if and only if* $\#X \leqslant 2$)

Let $\langle X, d\rangle$ be a metric space. If $\#X \leqslant 1$, (d-CR11) is trivially satisfied.
Suppose that $\#X = 2$. Then $X = \{x, y\}$. Suppose that A and B are not empty, then we are in one of the four following situations:

- $A = B$: then $\exists a \in A\cap$ and $\mathtt{Inj}_d(A, B) = d(a, a)$.
- $A = \{x\}, B = \{y\}$: then $\mathtt{Inj}_d(A, B) = d(x, y)$.
- $A = \{x\}, B = \{x, y\}$: then $\mathtt{Inj}_d(A, B) = d(x, x) + \mathtt{Dmax}(X, d) = d(x, y)$.
- Symmetric cases.

Suppose that X has at least 3 elements x, y, z. Suppose (*wlog*) $d(x, y) \leqslant d(x, z)$. Then $\mathtt{Inj}_d(\{x\}, \{y, z\}) = d(x, y) + \mathtt{Dmax}(x, d) > d(x, y) \geqslant d(x, z)$.

□

Proof. ($\mathtt{Haus}_d$ *satisfies* (d-CR11))
By definition of `Haus`. □

Proof. ($\mathtt{Nij}_d$ *does not satisfy* (d-CR11))
See counter-example 2. □

Proof. ($\mathtt{PointSet}_d$ *does not satisfy* (d-CR11))
See counter-example 2. □

Proof. ($\mathtt{WSD}_d$ *does not satisfy* (d-CR11))
See counter-example 2. □

Proof. ($\mathtt{MaxSum}_d$ *does not satisfy* (d-CR11))
See counter-example 2. □

Proof. ($\mathtt{Max}_d$ *satisfies* (d-CR11))
By definition of `Max`.

□

Proof. ($\mathtt{Pairs}_d$ *does not satisfy* (d-CR11))
See counter-example 2. □

B.4.9 (d-CR17a)

Proof. ($\texttt{Drast}$ *satifies* (d-CR17a) *if and only if* $d = \texttt{Drast}$)

Let X be a set, $x, y \in X$. $\texttt{Drast}(\{x\},\{y\}) = 1 \iff x = y$ so $\texttt{Drast}(\{x\},\{y\}) = \texttt{Drast}(x, y)$
So $\texttt{Drast}$ satisfies ($\texttt{Drast}$-CR17a).
Reciprocally, let $\langle X, d\rangle$ be a metric space such that $\texttt{Drast}$ satifies (d-CR17a). Then for all $x, y \in X$, $\texttt{Drast}(\{x\},\{y\}) = d(x, y)$ and $\texttt{Drast}(\{x\},\{y\}) = 1 \iff x = y$, elseway $\texttt{Drast}(\{x\},\{y\}) = 0$. So $d = \texttt{Drast}$.

□

Proof. ($\texttt{Inj}_d$ *satisfies* (d-CR17a))
By definition of $\texttt{Inj}$. □

Proof. ($\texttt{Haus}_d$ *satisfies* (d-CR17a))
By definition of $\texttt{Haus}$. □

Proof. ($\texttt{Nij}_d$ *satisfies* (d-CR17a))
By definition of $\texttt{Nij}$. □

Proof. ($\texttt{PointSet}_d$ *satisfies:* $\forall x, y \in X, \texttt{PointSet}_d(\{x\},\{y\}) = 2d(x, y)$)
By definition of $\texttt{PointSet}$. □

Proof. ($\texttt{WSD}_d$ *satisfies:* $\forall x, y \in X, \texttt{PointSet}_d(\{x\},\{y\}) = 2d(x, y) + \texttt{Drast}(x, y)$)
By definition of $\texttt{WSD}$. □

Proof. ($\texttt{MaxSum}_d$ *satisfies* (d-CR17a))
By definition of $\texttt{MaxSum}$. □

Proof. ($\texttt{Max}_d$ *satisfies* (d-CR17a))
By definition of $\texttt{Max}$. □

Proof. ($\texttt{Pairs}_d$ *satisfies* (d-CR17a))
By definition of $\texttt{Pairs}$. □

B.4.10 (d-CR17b)

Proof. ($\texttt{Drast}$ *satisfies* (d-CR17b) *if and only if* $\texttt{Dmax}(X, d) = 1, ie.d = \texttt{Drast}$)
Let X be a set, and $x \in X$. $\texttt{Dmax}(X, \texttt{Drast}) = 1$. And:
$\texttt{Drast}(\{x\}, \emptyset) = 1 = \texttt{Dmax}(X, \texttt{Drast})$
So $\texttt{Drast}$ satisfies ($\texttt{Drast}$-CR17b).
Let $\langle X, d\rangle$ be a metric space. Suppose that $\texttt{Drast}$ satisfies (d-CR17b). Then:
Let $x \in X$, $\texttt{Drast}(\{x\}, \emptyset) = 1$ so $\texttt{Dmax}(X, d) = 1$ and d is a distance, so $d = \texttt{Drast}$.
So if $\texttt{Drast}$ satisfies (d-CR17b), then $d = \texttt{Drast}$.

□

Proof. ($\texttt{Inj}_d$ *satisfies* (d-CR17b))
By definition of $\texttt{Inj}$. □

Proof. ($\texttt{Haus}_d$ *satisfies* (d-CR17b))
By definition of $\texttt{Haus}$. □

Proof. ($\texttt{Nij}_d$ *does not satisfy* (d-CR17b))

If $X = \{1, 2, 3, 4\}$ and $\texttt{d}(x, y) \triangleq |x - y|$, then:
$\texttt{Dmax}(X, \texttt{d}) = 3$ and $\texttt{Nij}_\texttt{d}(\{1\}, 0) = 1 < \texttt{Dmax}(X, \texttt{d})$.

□

Proof. ($\texttt{PointSet}_d$ *satisfies* (d-CR17b))
By definition of $\texttt{PointSet}$. □

Proof. ($\texttt{WSD}_d$ *does not satisfy* (d-CR17b))
Same counter-example as for $\texttt{Nij}$. □

Proof. ($\texttt{MaxSum}_d$ *does not satisfy* (d-CR17b))
Same counter-example as for $\texttt{Nij}$. □

Proof. ($\texttt{Max}_d$ *satisfies* (d-CR17b))
By definition of $\texttt{Max}$. □

Proof. ($\texttt{Pairs}_d$ *does not satisfy* (d-CR17b) *and satisfies:* $\forall A \subseteq X, \texttt{Pairs}_d(A, \emptyset) = 0$)
By definition of $\texttt{Pairs}$.

□

B.4.11 (d-CR12)

Proof. ($\texttt{Sum}$ *satisfies* (d-CR12$_n$) *for all n corresponding to an index in the cartesian product*)
By definition of $\texttt{Sum}$.

□

B.5 The case of S5 Kripke models

B.5.1 Definition

Several specific classes of Kripke models can be defined, depending on the properties their accessibilty relation satisfies. Let's consider S5 Kripke models, defined in [Che80] as:

Definition 48. The *S5 Kripke models* are the Kripke models in which the following axiom schema are valid. For all propositions p, q, r of $\texttt{L}$:

- $\Box p \Rightarrow p$
- $\Box p \Rightarrow \Box\Box p$
- $p \Rightarrow \Box\Diamond p$

Proposition 23. *The S5 Kripke models are exactly the Kripke models where R is an equivalence relation.*

We can then represent a S5 Kripke model $\langle W, R, f\rangle$ in a different way, using the equivalence classes of R. Moreover, we consider that the valuation function of the considered models are injective, so that we can consider each world w as the valuation associated to it $f(w)$. This way, a S5 Kripke model becomes:

Definition 49. A *S5 Kripke state* is a pair $\langle w, W\rangle$ where W is a set of valuations on $\texttt{A}$ and w belongs to W.
A *S5 Kripke model* is a set M of S5 Kripke states, such that $\forall \langle w, W\rangle \in M, \forall w' \in W, \langle w', W\rangle \in M$ and $\forall \langle w, W\rangle, \langle w', W'\rangle \in M$ either $W = W'$, or $W \cap W' = \emptyset$.

We then define the truth value of a proposition in such models:

Definition 50. We define recursively the truth of a formula of $\texttt{L}$ in a state w of a model $\langle w, W\rangle$ by:

$$\begin{array}{lll} w \models \top & & \\ w \not\models \bot & & \\ w \models a & \text{if and only if} & w(a) = 1 \text{ when } a \in \texttt{A} \\ w \models \neg\varphi & \text{if and only if} & w \not\models \varphi \\ w \models \varphi \wedge \psi & \text{if and only if} & w \models \varphi \text{ and } w \models \psi \\ w \models \Box\varphi & \text{if and only if} & \forall w' \in W, w' \models \varphi \end{array}$$

A formula φ of $\texttt{L}$ is true in a model M if $\forall \langle w, W\rangle \in M, w \models \varphi$.

B.5.2 Pre-distances and distance on S5 Kripke models

Definition 51 ($\star$)**.** Let Ω be a set and $M \triangleq \langle W, R, f\rangle$ be a Kripke model. We say that M *takes its values in* Ω if $W \subseteq \Omega$.
We say that M is *finite* is W is finite.

S5 Kripke models correspond to the specific case where R is an equivalence relation. Each equivalence class can be seen as the point of view of an agent. In the case where there is a single agent, there is also a single equivalence class and R is the universal relation. Then a S5 Kripke model with an unique equivalence class can be represented as a pair $\langle W, f\rangle$. If f is injective, we can replace W by $f(W)$. In that very specific case, a S5 Kripke model can be represented by the set $f(W)$. A state of such model is then of the form $\langle f(w), f(W)\rangle$ or simply $f(w)$.
In this section, we only consider the case where the interpretation functions of the models are injectives. The more general case is treated in Section 4.

B.5.3 Specific case of a single agent

In the specific case of S5 Kripke models with a single equivalence class and an injective interpretation function, each model $\langle W, f\rangle$ can be represented by the set $f(W)$.

Proposition 24 ($\star$)**.** *Let Ω be a set, d be a distance on Ω, let S be a set-predistance function, $S(d)$ is a pre-distance between finite S5 Kripke models with a single equivalence class and taking their values in Ω.*
Besides, if S is a set-distance function, then $S(d)$ is a distance.

A distance on Ω always exits, for example with `Drast`.

Proposition 25 ($\star$)**.** `Delta` *is a pre-distance between finite S5 Kripke models with a single equivalence class.*

B.5.4 General case

Lemma 5 ($\star$)**.** *Let S be a set-predistance function, T be a tuple-distance and d be a distance on* $\texttt{Val}(\texttt{A})$. *$T(d, S(d))$ is a pre-distance between states of S5 Kripke models.*
Besides, if F is a set-distance function, then $T(d, S(d))$ is a distance between states of S5 Kripke models.

Corollary 3 ($\star$)**.** $\texttt{Sum}(\texttt{Ham}, \texttt{Haus}(\texttt{Ham}))$ *and* $\texttt{Sum}(\texttt{Ham}, \texttt{Inj}(\texttt{Ham}))$ *are distances between states of S5 Kripke models.*

Proposition 26 ($\star$)**.** *Let S_1 and S_2 be two set-predistance functions, T be a tuple-distance and d be a distance on* $\texttt{Val}(\texttt{A})$. *$S_2(T(d, S_1(d)))$ is a pre-distance between S5 Kripke models.*
Besides, if S_1 and S_2 are set-distances functions, then $S_2(T(d, S_1(d)))$ is a distance between S5 Kripke models.

Corollary 4 ($\star$)**.** $\texttt{Haus}(\texttt{Sum}(\texttt{Ham}, \texttt{Haus}(\texttt{Ham})))$, $\texttt{Inj}(\texttt{Sum}(\texttt{Ham}, \texttt{Inj}(\texttt{Ham})))$, $\texttt{Inj}(\texttt{Sum}(\texttt{Ham}, \texttt{Haus}(\texttt{Ham})))$ *and* $\texttt{HausSum}(\texttt{Ham}, \texttt{Inj}(\texttt{Ham}))$ *are distances between S5 Kripke models.*

It is also possible to use `Delta` in one of the steps of the construction, or any distance between sets. For example, since a S5 Kripke model is a set of states:

Proposition 27 ($\star$)**.** `Delta` *is a distance between S5 Kripke models.*

Or we can also use delta on the equivalence classes of the states:

Proposition 28 ($\star$)**.** $\texttt{Haus}(\texttt{Sum}(\texttt{Ham}, \texttt{Delta}))$ *and* $\texttt{Inj}(\texttt{Sum}(\texttt{Ham}, \texttt{Delta}))$ *are distances between S5 Kripke models.*

The same goes by replacing `Delta` by `Bin`.

More generally:

Proposition 29 ($\star$)**.** *Let S be a set-predistance function, D be a pre-distance between sets of valuations, T be a tuple-distance and d be a distance on* $\texttt{Val}(\texttt{A})$*. $S(T(d, D))$ is a pre-distance between S5 Kripke models.*
Besides, if S is a set-distance function and D is a distance, then $S(T(d, D))$ is a distance between S5 Kripke models.

Using the pre-distances defined in Section 3.1.4, we can define lots of other pre-distances. They will not be distances but they can have very interresting properties, which are expressed by the axioms they satisfy. It is also perfectly possible to use other pre-distances that are not defined in this document as constructing tools.

The final question to be answered is to know which axioms are the most desirable for our pre-distance to satisfy, in order to choose which combination to use.

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