Bachelor internship
Distances between Kripke models for distance-based belief revision

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Outline

1. Introduction
   - Modal logic
   - Belief revision
   - Distance-based belief revision
2. Requirements on distances
   - Axioms on pre-distances
   - Impossibility result
3. Pre-distances and distances
   - Candidates
   - Axiomatic study of the candidates
4. Application to belief revision in modal logic
5. Conclusion and perspectives

Introduction

$\star$: propositions that I proved and definitions that I introduced

$\diamond$: already proved in the literature

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Modal logic

Language: $\mathtt{L}$ generated by $\langle\ \mathtt{A}, \wedge, \neg, \Box, \top, \bot\ \rangle$ where $\mathtt{A}$ is a finite set of atomic propositions.

$\Box$: "necessary"
$\Diamond$: "possible"

$\Diamond\varphi$ is equivalent to $\neg\Box\neg\varphi$.

Modal logic

Definition

A *valuation* on $\mathtt{A}$ is a function from $\mathtt{A}$ to $\{0,1\}$. The set of all valuations on $\mathtt{A}$ is noted by $\mathtt{Val}(\mathtt{A}) \triangleq \{0,1\}^{\mathtt{A}}$.

Definition

A *Kripke model* is a triple $\langle W, R, f\rangle$ where:

- W is a non empty set. Its elements are called "worlds".
- R is a relation on W called *accessibility relation*.
- $f : W \to \mathtt{Val}(\mathtt{A})$ is called *interpretation function* or *labelling function*

Modal logic

Definition

We define recursively the truth of a formula of $\mathtt{L}$ in a world w of a model $\langle W, R, f \rangle$ by:

$w \models \top$, $w \not\models \bot$		
$w \models a$	if and only if	$f(w)(a) = 1$ when $a \in \mathtt{A}$
$w \models \neg\varphi$	if and only if	$w \not\models \varphi$
$w \models \varphi \wedge \psi$	if and only if	$w \models \varphi$ and $w \models \psi$
$w \models \Box\varphi$	if and only if	$\forall w' \in W, wRw' \Rightarrow w' \models \varphi$

A formula φ of $\mathtt{L}$ is true in a model $\langle W, R, f \rangle$ if $\forall w \in W, w \models \varphi$.

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Belief revision

My favorit color is red

The color of my car is my favorit color

My car broke down

↓

A red car broke down

Belief revision

My favorit color is red

The color of my car is my favorit color

My car broke down

↓

~~A red car broke down~~

Belief revision

My favorit color is red (?)

The color of my car is my favorit color (?)

My car broke down (?)

↑

~~A red car broke down~~

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Distance-based belief revision

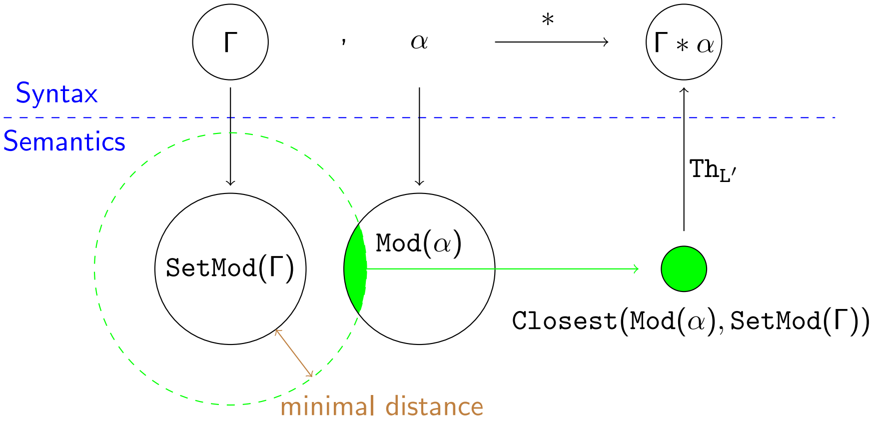

Figure: Process of the distance-based revision

Distance-based belief revision

Definition

We assume a language $\mathtt{L}'$ of classical logic, and X be a set of valuations on $\mathtt{L}'$, we define:

$\mathtt{Th}_{\mathtt{L}'}(X) \triangleq \{\alpha \in \mathtt{L}' \mid X \models \alpha\}$ (The set of all the formulas of $\mathtt{L}'$ satisfied by all valuations of X)

Let $\alpha \in \mathtt{L}'$, $\mathtt{Mod}(\alpha) \triangleq \{v \in \mathtt{Val}(\mathtt{A}) \mid v \models \alpha\}$. (The set of all valuations satisfying (modeling) α)

Let $\Gamma \in \mathtt{Pow}(\mathtt{L}'), \mathtt{SetMod}(\Gamma) \triangleq \{v \in \mathtt{Val}(\mathtt{A}) \mid v \models \Gamma\}$ (where $v \models \Gamma$ means $\forall \varphi \in \Gamma, v \models \varphi$).

Distance-based belief revision

Definition

A *revision operator* is a function $* : \mathtt{Pow}(\mathtt{L}') \times \mathtt{L}' \to \mathtt{Pow}(\mathtt{L}')$.

A *distance-based revision* is a revision operator such that, for all $\langle \Gamma, \alpha \rangle \in \mathtt{Pow}(\mathtt{L}') \times \mathtt{L}'$:

$$\Gamma * \alpha \triangleq \mathtt{Th}_{\mathtt{L}'}(\ \mathtt{Closest}(\mathtt{Mod}(\alpha), \mathtt{SetMod}(\Gamma))\ )$$

where $\mathtt{Closest}(\mathtt{Mod}(\alpha), \mathtt{SetMod}(\Gamma))$ is the set of the elements of $\mathtt{Mod}(\alpha)$ the closest to $\mathtt{SetMod}(\Gamma)$ for a distance *d* between valuations and set of valuations.

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Requirements on distances

Definition

Let X be a set. A *pre-distance* on X is a function from $X \times X$ to $[0, +\infty)$.

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Axioms on pre-distances

Axioms on any pre-distances

Definition ($\star$)

Let X be a set and F be a pre-distance on X. We define the following axioms:

(CR1) : (*Identity respecting*)
$\forall A, B \in X,\ F(A, B) = 0 \iff A = B$

(CR2) : (*Symmetric*)
$\forall A, B \in X,\ F(A, B) = F(B, A)$

(CR3) : (*Triangle inequality*)
$\forall A, B, C \in X,\ F(A, B) \leqslant F(A, C) + F(C, B)$

Definition

A *distance* is a pre-distance that satisfies (CR1), (CR2) and (CR3).

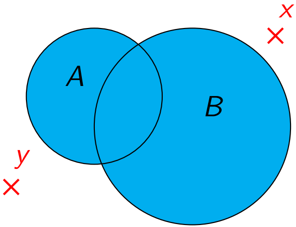
Axioms on pre-distances

Axioms on pre-distances between sets

Definition ($\star$)

Let X be a set of sets and F a pre-distance on X. We define the following axiom:

(CR6) : (*Stranger's equivalence*)
$\forall A, B \in X, \forall x, y \in \bigcup X,$
$\{x, y\} \cap (A \cup B) = \emptyset \Rightarrow F(A \cup \{x\}, B) = F(A \cup \{y\}, B)$



Axioms on pre-distances

Definition ($\star$)

Let X be a set of sets and F a pre-distance on X. We define the following axioms:

Let K be a real number,
($\mathbf{CR8}_K$) : (*K-countability*)
$\forall A, B \in X, \forall x \in \bigcup X,$
$\{x\} \cap (A \cup B) = \emptyset \Rightarrow F(A \cup \{x\}, B) = F(A, B) + K$

(CR16) : (*Inserting growth*)
$\forall A, B \in X, \forall x \in \bigcup X,$
$\{x\} \cap (A \cup B) = \emptyset \Rightarrow F(A \cup \{x\}, B) > F(A, B)$

Axioms on pre-distances

Axioms on pre-distances between subsets of a finite metric space

Definition ($\star$)

Let $\langle X, d\rangle$ be a finite metric space. We define the following axioms:
(d-CR4) : (*Elementary d-monotony*)

$$\forall (x, x', y, y') \in X^4, \forall A \subseteq X, \forall B \subseteq X,$$
$$(d(x, x') < d(y, y') \wedge \{x, y\} \cap A = \emptyset \wedge \{x', y'\} \cap B = \emptyset)$$
$$\Rightarrow F(A \cup \{x\}, B \cup \{x'\}) < F(A \cup \{y\}, B \cup \{y'\})$$

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Impossibility result

Definition

Let X be a set, we define the *Hamming distance* $\texttt{Ham}$ as the distance on $\texttt{Val}(X)$ such that $\forall v, w \in \texttt{Val}(X)$:
$\texttt{Ham}(v, w) \triangleq \#\{x \in X \mid v(x) \neq w(x)\}$

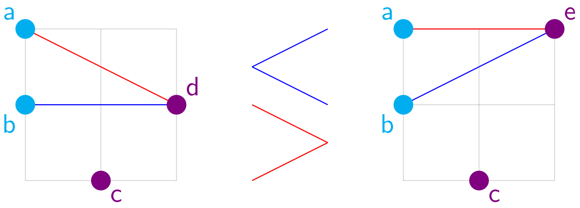
Proposition ($\star$)

($\texttt{Eucl}$-CR4) *cannot be satisfied (if* $\texttt{Eucl}$ *is the euclidian distance in a geometric space).* ($\texttt{Ham}$-CR4) *cannot be satisfied.*

Impossibility result

Idea of the proof:

$\texttt{Eucl}(a,d) > \texttt{Eucl}(a,e) \implies F(\{b,a\},\{c,d\}) > F(\{b,a\},\{c,e\})$
$\texttt{Eucl}(b,e) > \texttt{Eucl}(b,d) \implies F(\{a,b\},\{c,e\}) > F(\{a,b\},\{c,d\})$



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Candidates

Most general distance

Definition

Let X be a set. The *drastic distance* on X is the distance on X such that $\forall A, B \in X$: $\texttt{Drast}(A, B) \triangleq \begin{cases} 0 \text{ if } A = B \\ 1 \text{ otherwise} \end{cases}$

Candidates

Distances on cartesian products

Lemma ($\star$)

Let $n \in \mathbb{N}$, $(\langle X_i, d_i \rangle)_{i \in [\![1,n]\!]}$ *be* n *metric spaces. We note* $\mathbf{d} \triangleq (d_i)_{i \in [\![1,n]\!]}$ *and* $\mathbf{X} \triangleq X_1 \times \ldots \times X_n$, *then the function* $\mathtt{Sum}_{\mathbf{d}}$ *on* $\mathbf{X} \times \mathbf{X}$ *such that* $\forall \mathbf{x}, \mathbf{y} \in \mathbf{X}$*:*

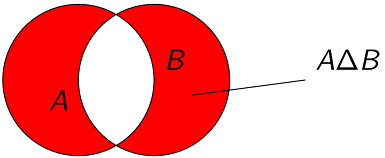
$$\mathtt{Sum}_{\mathbf{d}}(\mathbf{x}, \mathbf{y}) \triangleq \sum_{i=1}^{n} d_i(x_i, y_i) \text{ is a distance on } \mathbf{X}.$$

Candidates

Pre-distances between sets

Definition ($\diamond$)

Let X be a set of finite sets. We define $\texttt{Delta}(A, B) \triangleq \#(A \Delta B)$ where $A \Delta B$ is the *symmetric difference* between A and B (ie. $A \Delta B \triangleq (A \setminus B) \cup (B \setminus A)$).



Candidates

Pre-distances between sets

Proposition ($\diamond$)

`Delta` *is a distance on any set of finite sets.*

Proposition ($\star$)

Let X be a finite set, `Delta` *is the only pre-distance satisfying* (CR1), (CR2) *and* (CR8_1) *on* $\texttt{Pow}(X)$.

Proof: Double induction on the cardinality of the sets.

Candidates

Pre-distances between subsets of a metric space

Definition ($\star$)

Let $\langle X, d \rangle$ be a finite metric space, we define

$$\mathtt{Dmax}(X, d) \triangleq \max_{(x,y) \in X^2} d(x, y).$$

Candidates

Pre-distances between subsets of a metric space

Definition ($\star$)

Let $\langle X, d\rangle$ be a metric space. We define the following functions on $\texttt{Pow}(X) \times \texttt{Pow}(X)$, such that $\forall A, B \in \texttt{Pow}(X)$: (with $\min(\emptyset) = 0$)

$$\texttt{Inj}_d(A, B) \triangleq \min_{f:A\hookrightarrow B} \texttt{Dist}(d, A, B, f) + \min_{f:B\hookrightarrow A} \texttt{Dist}(d, B, A, f) + |\#A - \#B| \times \texttt{Dmax}(X, d)$$

where: $\texttt{Dist}(d, A, B, f) \triangleq \sum_{a\in A} d(a, f(a))$

Candidates

Pre-distances between subsets of a metric space

Definition ($\star$)

Let $\langle X, d\rangle$ be a metric space. We define the following functions on $\mathtt{Pow}(X) \times \mathtt{Pow}(X)$, such that $\forall A, B \in \mathtt{Pow}(X)$: (with $\min(\emptyset) = \mathtt{Dmax}(X, d)$)

$$\mathtt{PointSet}_d(A, B) \triangleq \sum_{x \in A} \mathtt{dist}(x, B, d) + \sum_{y \in B} \mathtt{dist}(y, A, d)$$

where $\mathtt{dist}(x, B, d) \triangleq \min\{d(x, y) | y \in A\}$

Proposition ($\star$)

Let $\langle X, d\rangle$ be a metric space, $\mathtt{Inj}_d$ is a distance between the subsets of X, and $\mathtt{PointSet}_d$ is not.

Candidates

Pre-distances between subsets of a metric space

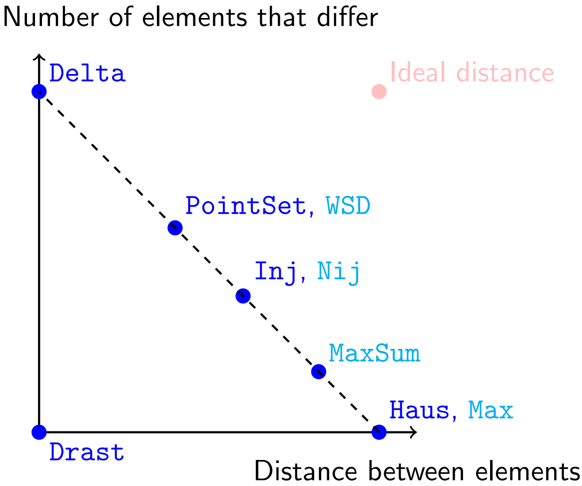


Figure: Expressivity of the pre-distances

Candidates

Distances between relations

Let X be a set.
A relation on X is a set of pairs of elements of X.
Let d be a distance on X.
Then $\mathtt{Sum}_{(d,d)}$ is a distance on $X \times X$.
Then $\mathtt{Haus}_{\mathtt{Sum}(d,d)}$ is a distance on $\mathtt{Pow}(X \times X)$, ie., a distance between relations on X.

Proposition (⋆)

Let $\langle X, d \rangle$ be a metric space, $\mathtt{Haus}_{\mathtt{Sum}(d,d)}$ and $\mathtt{Inj}_{\mathtt{Sum}(d,d)}$ are distances between relations on X.

The same can be done with other pre-distances.

Candidates

Distances between functions

Definition

Let X, Y be two sets. Let $f : X \to Y$ be a function, we define the *graph* of f as $\texttt{Graph}(f) \triangleq \{\langle x, f(x)\rangle \mid x \in X\}$.

Proposition ($\star$)

Let X be a finite set, $(f, g) \mapsto \texttt{Delta}(\texttt{Graph}(f), \texttt{Graph}(g))$ is a distance between functions from X to any other set.

Proposition ($\star$)

Let $\langle X, d\rangle$ and $\langle Y, d'\rangle$ be two metric spaces,
$(f, g) \mapsto \texttt{Haus}_{\texttt{Sum}(d,d')}(\texttt{Graph}(f), \texttt{Graph}(g))$ and
$(f, g) \mapsto \texttt{Inj}_{\texttt{Sum}(d,d')}(\texttt{Graph}(f), \texttt{Graph}(g))$ are distances between functions from X to Y.

Candidates

Distances between functions

Definition (⋆)

Let X be a set and $\langle Y, d \rangle$ be a finite metric space, and let f and g be two functions from X to Y. We define:
$$\mathtt{ExtHam}_d(f, g) \triangleq \sum_{x \in X} d(f(x), g(x)).$$

$\mathtt{Ham}$ then is exactly $\mathtt{ExtHam}_{\mathtt{Drast}}$ applied to valuations.

Proposition (⋆)

Let X be a finite set and $\langle Y, d \rangle$ be a finite metric space, $\mathtt{ExtHam}_d$ is a distance on Y^X.

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Axiomatic study of the candidates

		Pre-distances		Pre-distances Between sets		Pre-distances between subsets of a metric space								Pre-distances Between Tuples	
		Drast	Plus(d1,d2)	Delta	Bin	Inj(d)	Haus(d)	Nij(d)	PoinSet(d)	WSD(d)	MaxSum(d)	Max(d)	Pairs(d)	Sum(d,...)	Names of the axioms :
Axioms on Pre-distances	CR1	V	V	V	V	V	V	V	V	V	X	X	X****	V	*Identity respecting*
	CR2	V	V	V	V	V	V	V	V	V	V	V	V	V	*Symmetry*
	CR3	V	V	V	V	V	V	X***	X	X	X	V	X	V	*Triangle inequality*
	D-CR9	D	d, d1, d2	D		D								D	*D-monotony*
Axioms on Pre-distances Between sets	CR5	V		V	X	V	X	V	X	X	X	X	X		*Subset equivalence*
	CR6	V		V	X	X	X	X	X	X	X	X	X		*Stranger equivalence*
	CR7 far	X		X	X	X	X	X	X	X		X			*Far decomposability*
	CR7 str	X	d1,d2	X	X	X	X	X	X	X	V	X	V		*Stranger decomposability*
	CR8 K	X		V	X	X	X	X	X	X	X	X	X		*K-denombrability*
	CR9	X		V	X	X	X	X	X	X	X	X	X		*Delta-monotony*
	CR16	X		V	X	X	X	X	X	V	X	X	X		*Inserting growth*
	CR18	V		V	V		X			V	X	X	X		*Intersection indifference*
Axioms on Pre-distances Between substes Of a metric space	d-CR4 far	d	d,d1,d2			V									*Far elementary d-monotony*
	d-CR4 far'											V			*Far elementary d-monotony bis*
	d-CR10			X	d	X	X	X	X	X	V	X			*Global d-monotony*
	d-CR10'			X							V				*Global injective d-monotony*
	d-CR11					X*	V	X	X	X	X	V	X		*d-represented*
	d-CR17a	S*(7)				V	V	V	V*(5)	X*(6)	V	V	V		d-singleton fidelity
	d-CR17b	S*(8)				V	V	X	V	X	X	V	X*(9)		d-void maximality
Pre-distance Between tuples	d-CR12		d, d1, d2											V	*d-left/right/n-monotony*
	d-CR12S													d	*Strong d-left/right/n-monotony*
Between elements of Type (element,set)	D-CR13	X	d1,d2												*Union D-monotony*
	CR14	X													*Membership distinction*

Figure: Axiomatic study of the introduced pre-distances

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Application to belief revision in modal logic

Corollary ($\star$)

The following functions are distances between finite Kripke models:

$\mathtt{Sum}(\mathtt{Haus}(\mathtt{Drast}), \mathtt{RDF}(\mathtt{Haus}, \mathtt{Sum}, \mathtt{Drast}), \mathtt{FDF}(\mathtt{Haus}, \mathtt{Sum}, \mathtt{Drast}, \mathtt{Ham}))$

$\mathtt{Sum}(\mathtt{Inj}(\mathtt{Drast}), \mathtt{RDF}(\mathtt{Inj}, \mathtt{Sum}, \mathtt{Drast}), \mathtt{FDF}(\mathtt{Inj}, \mathtt{Sum}, \mathtt{Drast}, \mathtt{Ham}))$

$\mathtt{Sum}(\mathtt{Haus}(\mathtt{Drast}), \mathtt{RDF}(\mathtt{Haus}, \mathtt{Sum}, \mathtt{Drast}), \mathtt{ExtHam}_{\mathtt{Ham}})$

$\mathtt{Sum}(\mathtt{Haus}(\mathtt{Drast}), \mathtt{RDF}(\mathtt{Inj}, \mathtt{Sum}, \mathtt{Drast}), \mathtt{Delta})$

$\mathtt{Sum}(\mathtt{Delta}, \mathtt{Delta}, \mathtt{Delta})$

and so on...

If there is a distance on Ω, $\mathtt{Drast}$ can be replaced by this distance.

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Conclusion and perspectives

What I did during this internship

- Goal of the internship: find a distance between Kipke models
- Serveral distances found and more studied
- Axiomatic study of the found distances
- General methods for the construction of distances
- Characterization of `Delta` and `Pairs`
- Impossibility result on (CR4)
- An article in progress

Conclusion and perspectives

What next ?

- Pertinence of each axiom
- Application to real cases
- The difficulty of the translation between syntax and semantics

Thank you for your attention.