Math club - Session 1: Logic, variables and functions
An Introduction to the Language of Mathematics

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1 Basic logic and sets

1.1 Basic logic symbols

We write $A, B, C, ...$ for propositions. For example A could represent "I like blue." or "x=2".

Symbol	Meaning (en)	Meaning (fr)
$A \Rightarrow B$	A implies B	A implique B
$A \Leftarrow B$	A is implied by B	A est impliqué par B
$A \iff B$	A is equivalent to B, A if and only if B (A iff B)	A est équivalent à B, A si et seulement si B (A ssi B)
$A \not\Rightarrow B$	A does not imply B	A n'implique pas B
$\neg A$	not A	non A
$\exists x \mid A$	There exists x such that A is true (A depending on x)	Il existe x tel que A est vrai (A dépendant de x)
$\forall x.A$	For all x, A is true (A depending on x)	Pour tout x, A est vrai (A dépendant de x)

Example: Contraposition

Proposition 1. *If $A \Rightarrow B$ then $\neg B \Rightarrow \neg A$.*

Proof. (Proof by contradiction)
Let $A \Rightarrow B$ be true. Let $\neg B$ be true, suppose A. Then, since $A \Rightarrow B$, we know that B is also true. Contradiction. So we have $\neg A$. □

Remark: In fact we don't say "A is true" but just "A".

Exercise 1: True or false? If $A \Rightarrow B$ and $B \Rightarrow C$ then:

1) $B \Rightarrow A$ 2) $C \Rightarrow A$ 3) $A \Rightarrow C$

Remark: This is called *transitivity of implication.*

Exercise 2: True or false? If $A \Rightarrow B$ and $B \Rightarrow A$ and $\neg B$, then:

1) $A \Rightarrow \neg B$ 2) $A \Rightarrow \neg A$ 3) $A \iff B$ 4) $\neg A$

1.2 Sets

A *set* is a group of elements, it can contain anything. In sets, the elements occure at most one time and are not ordered. For example $\{1, 2, 3\}$ is the same set as $\{2, 1, 3\}$ and $\{1, 1, 1\}$ is not a set. $\{\star, \triangle, \square, \diamond\}$ is also a set. Set can be finite or infinite. For example $\mathbb{N} = \{0, 1, 2, 3, 4, 5, ...\}$ is infinite.

Symbol	Meaning	Name (en)	Name (fr)
$\{1, a, \star, \pi\}$	Set containing $1, a, \star, \pi$	A set	Un ensemble
$\emptyset$	Set containing nothing	Empty set	Ensemble vide
$x \in S$	The element x is contained in the set S	x is in S	x est dans S
$x \notin S$	x is not in S		
$\mathbb{N}$	$\{0, 1, 2, 3, ...\}$	Natural numbers	Entiers naturels
$\mathbb{Z}$	$\{..., -2, -1, 0, 1, 2, ...\}$	Integers	Entiers relatifs
$\mathbb{Q}$	Numbers that can be written as $\frac{n}{m}$ with $n, m \in \mathbb{Z}$	Rational numbers	Nombres rationnels
$\mathbb{R}$	All numbers: $-2, 8.99999..., \pi, 10^{100}$	Real numbers	Nombres réels
$A \subset B$	Set A is included in set B	A included in B	A inclus dans B
$A \supset B$	Set A contains set B	A contains B	A contient B
$A \not\subset B$	Set A is not included in set B		
$A \cup B$	Set containing the elements of A and those of B	A union B	A union B
$A \cap B$	Set containing the elements that are both in A and B	A intersect B	A inter B
$A \setminus B$	Set containing the elements of A without those of B	A minus B	A privé de B
$(1, 2, 3)$	Tuple containing 1,2,3 in that order	tuple	n-uplet

Remark: Tuples are not sets, they can have several times the same element occurring and the order matters. Usually, they are finite. Tuples of size two are *pairs* and tuples of size three are *triples*. For example $(2, 2)$ is a valid pair, but $\{2, 2\}$ is not a valid set, and $(1, 2) \neq (2, 1)$.

Exercise 3: True or false?

- $\exists n \in \mathbb{N} \mid 1 + n = 5$
- $\exists n \in \mathbb{N} \mid 1 + n = 0$
- $\exists n \in \mathbb{Z} \mid 1 + n = 0$
- $\forall n \in \mathbb{N} \mid 1 + n = 0$
- $\forall n \in \mathbb{N} \mid 1 + n \in \mathbb{N}$

Exercise 4: Let $A = \{1, 2, 3, 7\}$, $B = \{1\}$ and $C = \{\star, \triangle, \square\}$.

- Do we have $A \subset B$?
- $A \cap C = ?$
- $A \cap B = ?$
- $B \cup C = ?$
- $A + B = ?$
- $A \setminus B = ?$
- Do we have $1 \in A$?
- Do we have $A \setminus C$?
- Do we have $1 \in C$?
- Do we have $1 \in A \cap B$?
- Do we have $2 \in A \cup C$?

2 Variables and functions

2.1 Variables

Variables are mathematical objects which have a name but whose value can be unknown. They can have different natures: numbers, integers, sets, functions, and so on.

For example in the equation $x = y + 1$, x and y are variables and we know that they are real numbers. In the proposition $x \in \{\Box, \triangle, \diamond\}$, we know that x is a geometrical symbol. In $A \subset B$, A and B are set variables.

Sometimes we use variables t represent a value that we want to calculate, but sometime we only need partial informations about this variable.

Example: My friend said she can come with me to a trip only if it costs at most 200€. We define the variable p which represents the price of the trip that I want to organize (price per person). Here, we only need to know if $p \leq 200$.

Remark: We need to know if $p \leq 200$ and not $p < 200$, why ?

Exercise 5: Following the previous example, translate the following sentences in math, using variables when needed. Then tell me if my friend will be able to come to my trip.

- Train will cost between 30 and 60€.
- Hotel cost less than 90€.
- We will eat at restaurant three times for 10€ each time.
- We won't spend money except for train, hotel and restaurant.

2.2 Functions

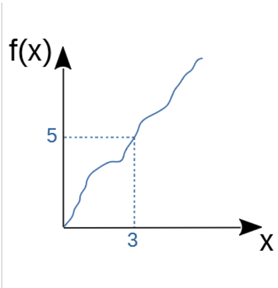
Functions are mathematical objects which take elements as input and associate an output to each input. This is a very computational vision, a true mathematician would rather say that a function is a set of arrows, going from the input to the output. Let's see how the many possible representations of functions actually describe the same object.

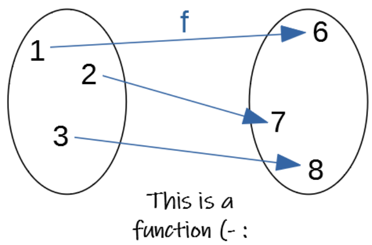
The full formal notation is:

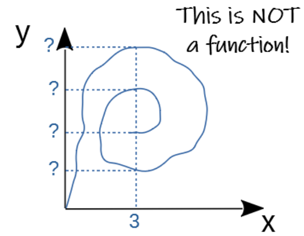
$$f : \left| \begin{array}{ccc} A & \longrightarrow & B \\ x & \longmapsto & f(x) \end{array} \right.$$

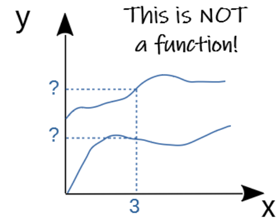
Which means that f is a function from A to B (*ie.* inputs are all the elements of the set A and outputs are in B), which associates to each x in A the value $f(x)$.

For example: $f : \left| \begin{array}{ccc} \mathbb{N} & \longrightarrow & \mathbb{Z} \\ n & \longmapsto & 1 - n \end{array} \right.$

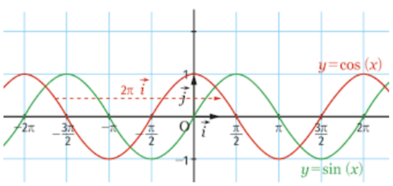


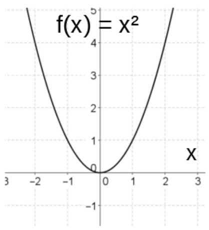


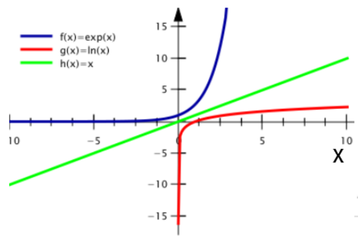


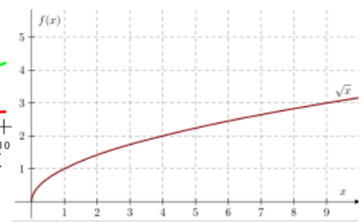


Some classical functions:







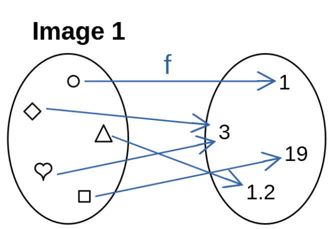


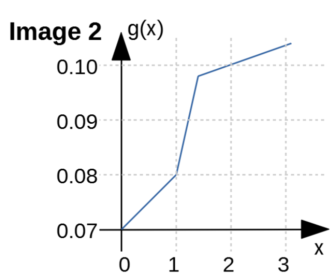
A point in full math language: Let $f : A \rightarrow B$ be a function, the *inverse function* of f is the unique function f^{-1} such that: $\forall a \in A.\ f^{-1}(f(a)) = a$. It exists only if f is bijective (*ie.* surjective: $\forall b \in B, \exists a \in A. f(a) = b$ and injective: $\forall a, a' \in A, f(a) = f(a') \Rightarrow a = a'$).

3 Exercises

Exercise 6:

- We define the function f as $f(x) = 4x + 3$. $f(5) = ?$
- We define g as $g(x) = ax + b$ and we know that $g(0) = 5$ and $f(3) = 11$. Find the values of a and b.
- Give a function f such that $f(3) = 0$.
- Give a function g such that $f(0) = 1$ and $f(2) = 3$.
- With image 1: $f(\square) = ?$ With image 2: $g(2) = ?$





Exercise 7: Calculate the value of variables, when possible x)

- $x + 1 = 2$
- $x + y = 2$
- $3x + 5 = 3$
- $x + y = 2$ and $x - y = 0$
- $a + b = 6$ and $a = 2b$
- $3x + 4 = 5x + 9$
- $\frac{x}{5} = \frac{3}{8}$
- $x = \frac{9}{y}$ and $x, y \in \mathbb{N}$